ISSN 1028-3358, Doklady Physics, 2011, Vol. 56, No. 9, pp. 494–497. © Pleiades Publishing, Ltd., 2011.
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MECHANICS

Eigenvalue Problems for the Generalized Orr–Sommerfeld Equation in the Theory of Hydrodynamic Stability

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Presented by Academician V.A. Sadovnichii April 6, 2011

Received April 8, 2011

DOI: 10.1134/S1028335811090023

For uncompressed steady-state flows of non-Newtonian fluids with tensor linear determining relations and an arbitrary nonlinear dependence of intensity of stresses on the intensity of the deformation rates (of the arbitrary rheological curve), the eigenvalue problem generalizing the classical Orr–Sommerfeld problem and modeling the shear stability of such flows relative to small perturbations is very important.

We present below one of the possible similar statements and develop the method of integral relations as applied to it. For a Newtonian fluid, this method leads to lower evaluations of the critical values of a unique dimensionless parameter taking into account the properties of the medium, specifically the Reynolds number; on the contrary, three independent parameters of the rheological curve enter the sufficient evaluations of stability in the case under consideration.

1. In mathematical physics and the linearized theory of hydrodynamic stability, the eigenvalue problem for the Orr–Sommerfeld equation

$$\frac{1}{\mathrm{Re}}(\varphi^{\mathrm{IV}} - 2s^2\varphi'' + s^4\varphi) = (\alpha + isv^\circ)(\varphi'' - s^2\varphi) - isv^{\circ\prime\prime}\varphi \qquad (1)$$

relative to the complex-signed function $\varphi(x)$ with adherence conditions to flat boundaries

$$\varphi(0) = \varphi(1) = 0, \quad \varphi'(0) = \varphi'(1) = 0, \qquad (2)$$

called the Orr–Sommerfeld problem, is classic [1–8]. It simulates the temporal development of the flat pattern of perturbations imposed on the steady-state shear flow of the Newtonian viscous fluid in the layer $\Omega = \{(x_1, x_2){:}\ -\infty < x_1 < \infty,\ 0 < x_2 < 1\}$ with the kinematics

$$v_1^\circ = v^\circ(x) \in C^2, \quad v_2^\circ \equiv 0 \ \ (x \equiv x_2). \qquad (3)$$

In (1), $\alpha = \alpha_* + i\alpha_{**}$ is the spectral parameter (the complex frequency), and $s > 0$ is the wave perturbation number along axis x_1. The function $\varphi(x)$ is associated with the variation in the current function $\delta\psi$ $\left(\delta v_1 = \frac{\partial\delta\psi}{\partial x},\ v_2 = -\frac{\partial\delta\psi}{\partial x_1}\right)$ by the relationship

$$\delta\psi(x_1, x, t) = \varphi(x)e^{isx_1 + \alpha t}. \qquad (4)$$

The sign of variation δ at φ is omitted for brevity.

The Reynolds number Re in (1) is designed by the so-called dynamic viscosity coefficient μ in the dimensionless basis usual for a class of problems $\{\rho, V, h\}$, where ρ is the constant fluid density and V and h are the characteristic velocity and thickness of the layer. In all formulas, the dimensionless form of writing is accepted.

It is known that the Newtonian fluid, for which problem (1), (2) is formulated, is described by the linear relation of quadratic invariants $T = \sqrt{\frac{s_{mn}s_{mn}}{2}}$ and $U = \sqrt{2v_{mn}v_{mn}}$ of the stress deviator $\underset{\sim}{s}$ and deformation rate tensor $\underset{\sim}{v} = \mathrm{Def}\mathbf{v}$:

$$T(U) = \frac{U}{\mathrm{Re}}, \quad T^{\boldsymbol\cdot} \equiv \frac{dT}{dU} = \frac{1}{\mathrm{Re}} = \mathrm{const}. \qquad (5)$$

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The tensors $\underset{\sim}{s}$ and $\underset{\sim}{v}$ are coaxial:

$$s_{kl} = \frac{2T(U)}{U} v_{kl}, \tag{6}$$

i.e., the determined relations are tensorially linear or quasi-linear. For a Newtonian fluid, the coaxiality coefficient 2/Re is independent of U, which additionally means the physical linearity of the medium.

Let us consider the generalizations of problem (1), (2) to uncompressible flows on a non-Newtonian fluid with tensorially linear determining relations (6) but with the arbitrary experimentally determined dependence $T(U)$ called the rheological or universal curve. We assume that this dependence satisfies the requirements

$$T(0) = 0, \quad T^{\cdot}\big|_{U \geq 0} > 0. \tag{7}$$

As the unperturbed flow, the parameters of which are marked below with the index "°", as before we select a unidimensional steady-state shift (3), for which

$$U^\circ = 2\left|v_{12}^\circ\right| = |v^{\circ\prime}|, \quad s_{12}^\circ = T_\circ \operatorname{sgn} v_{12}^\circ,$$
$$T_\circ \equiv T(U^\circ(x)), \quad T_\circ^{\cdot} \equiv \frac{dT}{dU}(U^\circ(x)). \tag{8}$$

The generalized Orr–Sommerfeld equation [9–12]

$$\left(\frac{d^2}{dx^2} + s^2\right)\left[T_\circ^{\cdot}(\varphi'' + s^2\varphi)\right] - 4s^2\left(\frac{T_\circ}{U^\circ}\varphi'\right)'$$
$$= (\alpha + isv^\circ)(\varphi'' - s^2\varphi) - isv^{\circ\prime\prime}\varphi \tag{9}$$

can be derived in general by the method standard in the theory of hydrodynamic stability. Damping of perturbations is equivalent to the negativity of real parts of all eigenfrequencies: $\alpha_{j*}(s) < 0$ ($j \geq 1$) at any $s > 0$.

2. One of the approaches in the analytical investigation of problem (9), (2) is acquiring sufficient integral (energy) evaluations of stability. In the literature, this approach has the common name of the method of integral relations.

Let

$$\varphi(x) \in H_2, \quad \|\varphi\|^2 = \int |\varphi''|^2 dx. \tag{10}$$

Here and below, the integration is performed over x from zero to unity; in the same limits, we take the upper and lower faces of the functions with respect to x. Let us additionally multiply both parts of (9) by $\bar{\varphi}$ and integrate going to quadratic functionals:

$$L[\varphi; s] \equiv \int T_\circ^{\cdot}|\varphi''|^2 dx + 2s^2\int\left(\frac{2T_\circ}{U^\circ} - T_\circ^{\cdot}\right)|\varphi'|^2 dx$$
$$+ s^2\int\left[\left(T_\circ^{\cdot}\right)'' + s^2 T_\circ^{\cdot}\right]|\varphi|^2 dx$$
$$= -\alpha(I_1^2 + s^2 I_0^2) - is\int v^{\circ\prime}\varphi'\bar{\varphi}dx \tag{11}$$
$$- is\int[v^{\circ\prime\prime}|\varphi|^2 + v^\circ(|\varphi'|^2 + s^2|\varphi|^2)]dx,$$
$$I_n^2 = \int|\varphi^{(n)}|^2 dx, \quad n = 0, 1, 2.$$

Writing equality (11) for real parts, we acquire

$$\alpha_* = \frac{1}{I_1^2 + s^2 I_0^2}\left(s\int v^{\circ\prime}(\varphi'\bar{\varphi})_{**}dx - L[\varphi; s]\right). \tag{12}$$

The first parenthesized summand is evaluated from above using the Schwartz inequality in H_2:

$$\alpha_* \leq \frac{qsI_1I_0}{I_1^2 + s^2 I_0^2} - \Lambda \leq \frac{q}{2} - \Lambda, \tag{13}$$

$$q = \sup|v^{\circ\prime}|, \quad \Lambda[\varphi; s] = \frac{L[\varphi; s]}{I_1^2 + s^2 I_0^2}. \tag{14}$$

Thus, the problem is reduced to finding the absolute minimum of $\Lambda_{\min}$ over s of the functional $\Lambda[\varphi; s]$ for class of functions (10) with boundary conditions (2). If this minimum exists and is positive, the sufficient requirement of stability of the main shear with profile $v^\circ(x)$ is inequality

$$q < 2\Lambda_{\min} \tag{15}$$

following from (13).

3. For the Newtonian fluids, the lower evaluation of the critical Reynolds number Re* follows from inequality (15). To the contrary, for the non-Newtonian fluids, (15) involves the parameters of the entire rheological curve $T(U)$ rather than the single Re number. Let us introduce these parameters, namely, Re_1, a, and b into consideration:

$$\frac{1}{\mathrm{Re}_1} = \inf T_\circ^{\cdot}(x), \quad \frac{\pi^2 a}{\mathrm{Re}_1} = \inf\left(T_\circ^{\cdot}\right)''(x),$$
$$\frac{b}{\mathrm{Re}_1} = 2\inf\left(\frac{2T_\circ}{U^\circ} - T_\circ^{\cdot}\right). \tag{16}$$

By virtue of requirements (7) on the dependence $T(U)$, the quantity Re_1 or the upper boundary of the Reynolds numbers constructed by the tangential viscosities is positive.

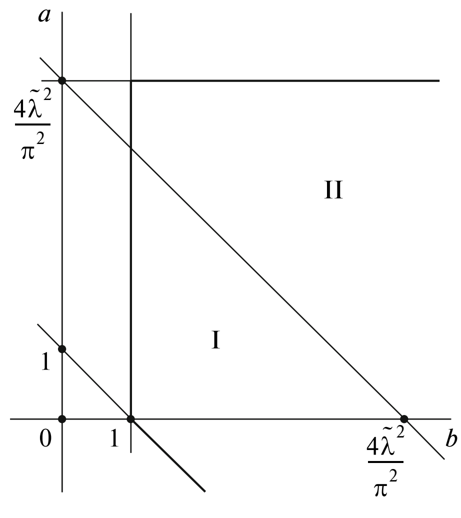

Fig. 1.

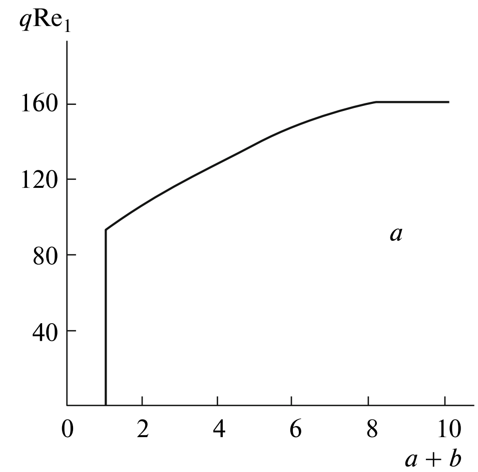

Fig. 2.

Let us further use the Fridrichs inequalities for functions (10) satisfying conditions (2):

$$I_1^2 > \pi^2 I_0^2, \quad \int T_\circ^{\cdot} |\varphi''|^2 dx \ge \frac{I_2^2}{\text{Re}_1} \ge \frac{4\tilde{\lambda}^2 I_1^2}{\text{Re}_1}, \qquad (17)$$
$$4\tilde{\lambda}^2 \approx 80.75,$$

where $\tilde{\lambda} \approx 4.493$ is the smallest positive root of the equation $\tan\lambda = \lambda$. The first of inequalities (17) can be deliberately enhanced because $I_1^2 = \pi^2 I_0^2$ only for functions $\varphi(x) = \varphi_0 \sin\pi x$, for which $\varphi'(0) \neq 0$ and $\varphi'(1) \neq 0$.

Allowing for (17), the functional $\Lambda[\varphi; s]$ (14) admits the evaluation from below:

$$\Lambda[\varphi; s] \ge \frac{1}{\text{Re}_1}\left[\frac{4\tilde{\lambda}^2 - \pi^2 a + (b-1)s^2}{1 + \frac{I_0^2}{I_1^2}s^2} + s^2 + \pi^2 a\right]. \qquad (18)$$

The minimum over s in the right part of (18) depends on the parameters of the curve $T(U)$ introduced in (16). Let us give its values in regions I and II on the plane of parameters a and b (Fig. 1):

$$\text{I} = \left\{(a, b): b \ge 1,\ 1 \le a + b \le \frac{4\tilde{\lambda}^2}{\pi^2}\right\},$$

$$\text{II} = \left\{(a, b): b \ge 1,\ a < \frac{4\tilde{\lambda}^2}{\pi^2} < a + b\right\}, \qquad (19)$$

$$\Lambda_{\min}\text{Re}_1 = \begin{cases} \pi^2(a+b-2) + 2\pi\sqrt{4\tilde{\lambda}^2 - \pi^2(a+b-1)} & \text{in I}, \\ 4\tilde{\lambda}^2 & \text{in II}. \end{cases}$$

The continuity and smoothness of the evaluating expression take place upon the passage through the straight line $a + b = 4\tilde{\lambda}^2/\pi^2$.

Let us substitute expression (19) for $\Lambda_{\min}$ into sufficient stability condition (15) and come to a conclusion on the validity of the following theorem.

Theorem. If

$$q\text{Re}_1 < \begin{cases} 2\pi^2(a+b-2) + 4\pi\sqrt{4\tilde{\lambda}^2 - \pi^2(a+b-1)} & \text{in I}, \\ 8\tilde{\lambda}^2 & \text{in II}, \end{cases} \qquad (20)$$

then the real part of any eigenvalue α of problem (9), (2) is negative.

Thus, if $b \ge 1$, $a < \frac{4\tilde{\lambda}^2}{\pi^2}$, and parameters $a + b$ and $q\text{Re}_1$ belong to the region a in Fig. 2, then one-dimensional shift (3) is deliberately stable relative to small perturbations.

Particularly, for the unperturbed Couette flow with the profile $v^\circ(x) = x$, which occurs for any dependence $T(U)$, the characteristics of the main flow introduced above are as follows:

$$U^\circ \equiv 1, \quad \frac{T_\circ}{U^\circ} \equiv T(1) = \frac{1}{\text{Re}_2} = \text{const},$$

$$T_\circ^{\cdot} = \frac{dT}{dU}(1) = \frac{1}{\text{Re}_1} = \text{const},$$

$$q = 1, \quad a = 0, \quad b = 2(2c-1), \quad c = \frac{\mathrm{Re}_1}{\mathrm{Re}_2}.$$

A consequence of the theorem is the following suggestion. If the inequality

$$\mathrm{Re}_1 < \begin{cases} 8\pi^2(c-1) + 4\pi\sqrt{4\tilde{\lambda}^2 - \pi^2(4c-3)}, \\ \text{if } \frac{3}{4} \le c < c_1, \\ 8\tilde{\lambda}^2, \text{ if } c \ge c_1 = \frac{1}{2} + \frac{\tilde{\lambda}^2}{\pi^2} \approx 2.54, \end{cases} \tag{21}$$

is fulfilled, the Couette flow is deliberately stable.

4. In [13], analytical investigations of the limiting transition of the thermomechanical characteristics for one-dimensional shear flows of highly viscous fluids to the corresponding characteristics for the two-constant viscoplastic body with the positive yield point k (the Il'yushin–Bingham model) were fulfilled. The following regularizing scalar ratio was selected:

$$T = \frac{U}{\mathrm{Re}} + \frac{2k}{\pi}\arctan\frac{U}{2\beta}, \tag{22}$$

where $\beta > 0$ is the controlling parameter of the model. If $\beta \to 0$, the quadratic invariant T at any fixed $U > 0$ tends to $U/\mathrm{Re} + k$, i.e., to the value corresponding to the Bingham body. The trend of the curves $T(U)$ (22) to the limiting one is nonuniform over β; therefore, the revelation of the question of whether or not the evaluations of stability of viscoplastic flows follow from those acquired in our study with a limiting transition similar to $\beta \to 0$ in (22) is of great theoretical interest.

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Translated by N. Korovin