Quantification of the degree of irreversibility in terms of material parameters by using Ziegler's non-linear constitutive relation for the stress-velocity gradient relationship

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The non-trivial task of an engineer is to derive or at least to use a constitutive relation which leads to a solvable equation but still expresses the system behavior adequately, so that the solution for the motion is accurate. Often the system has solid-like or fluid-like behavior, and an appropriate relation is chosen. In the so-called Bingham-Ilyushin matter both regimes occur simultaneously. This can be expressed by using a tensor-linear constitutive relation with a nonlinear (viscosity) term and even be solved in a specific case of material parameters ($b = 0$). The analytical solution for the velocity distribution in a channel with moving walls is presented here. The velocity gradients lead to an increase in temperature. The internal friction due to the (nonlinear) viscosity contributes to the entropy production and accounts for the irreversibility of the process. The aim is to analyze the role of each material parameter in the entropy production which might be used as a measure of irreversibility of the process.

1 Introduction

Consider an open system defined in three-dimensional space described by orthonormal (CARTESIAN) coordinates $x_i = (x_1, x_2, x_3)$. It may exchange a conserved quantity, such as mass, linear momentum, energy with its exterior. By using the conservation laws we obtain

$$\dot{\rho} + \rho \frac{\partial v_i}{\partial x_i} = 0 \, , \; \rho \dot{v}_i - \frac{\partial \sigma_{ji}}{\partial x_j} = \rho f_i \, , \; \rho \dot{u} + \frac{\partial q_i}{\partial x_i} = \sigma_{ji} \frac{\partial v_i}{\partial x_j} + \rho r \, , \tag{1}$$

with mass density, ρ, velocities, v_i, internal energy density, u. For a two-dimensional channel filled with Bingham-Ilyushin matter, these local equations must hold. Assuming no body forces, f_i, no radiation supply, r, for a stationary two-dimensional (x_1, x_2) flow problem with the seminverse ansatz $v_i = (v_1(x_2), 0)$, $u = u(x_2)$, $T = T(x_2)$, the partial differential equations to be solved read

$$0 = 0 \, , \; -\frac{\partial \sigma_{ji}}{\partial x_j} = 0 \, , \; \frac{\partial q_i}{\partial x_i} = \sigma_{21} \frac{\partial v_1}{\partial x_2} \, , \tag{2}$$

as already introduced and discussed in [1]. By using the following constitutive equations for the flux terms

$$q_i = -\kappa \frac{\partial T}{\partial x_i} \, , \; \sigma_{ij} = -p\delta_{ij} + 2\left(a + \frac{k}{\pi \sqrt{d_{(2)}}} \arctan\left(\frac{1}{b} \sqrt{d_{(2)}} \right) \right) d_{ij} \, , \; d_{(2)} = \frac{1}{2} d_{ij} d_{ij} = d_{12} d_{12} = \left(\frac{1}{2} \frac{\mathrm{d}v_1}{\mathrm{d}x_2} \right)^2 \tag{3}$$

the velocity and temperature profiles can be found. The analytical solution for the channel flow with zero boundary velocities on the top and bottom, driven only by a pressure gradient $|p'|$, is given in [1] for the specific case $b = 0$. The motion of the matter is that of a fluid-flow near the boundaries and a rigid-plug-flow in the middle. This plug flows like a rigid body along with the fluid at its boundaries. It has no velocity gradients. This led us to define a critical pressure gradient $|p'_{\mathrm{cr}}|$ to specify the start of the plug-flow. Instead we can also define geometrically related terms $\xi = \frac{|p'_{\mathrm{cr}}|}{|p'|} = \frac{r_0}{R}$ and $\eta = \frac{r_{\mathrm{s}}}{r_0}$ where $\eta\xi$ relates to the center and 2ξ to the width of the plug (see in Fig. 1). By using normalized quantities

$$\bar{x} = \frac{x_2}{R} \, , \; \bar{v} = \frac{v_1}{v_0} \, , \; v_0 = \frac{|p'|R^2}{a} \, , \; \bar{\sigma} = \bar{v}' \mp \bar{k} \, , \; \bar{k} = \frac{k}{|p'|R} \, , \; \bar{\sigma}_{ij} = \frac{\sigma_{21}}{|p'|R} \, , \tag{4}$$

the velocity profile for moving boundaries can be determined from the balance of linear momentum in the form:

$$\bar{v}(\bar{x}) = \begin{cases} \frac{1}{2}\left(1 - \bar{x}^2\right) - \xi\left(1 + \eta\right)\left(1 - \bar{x}\right) + v_{\mathrm{top}} & , \quad \forall \bar{x} : \xi + \eta\xi \le \bar{x} \le 1 \\ \text{const.} & , \quad \forall \bar{x} : -\xi + \eta\xi \le \bar{x} \le \xi + \eta\xi \\ \frac{1}{2}\left(1 - \bar{x}^2\right) - \xi\left(1 - \eta\right)\left(1 + \bar{x}\right) + v_{\mathrm{bottom}} & , \quad \forall \bar{x} : -1 \le \bar{x} \le -\xi + \eta\xi \end{cases} \; . \tag{5}$$

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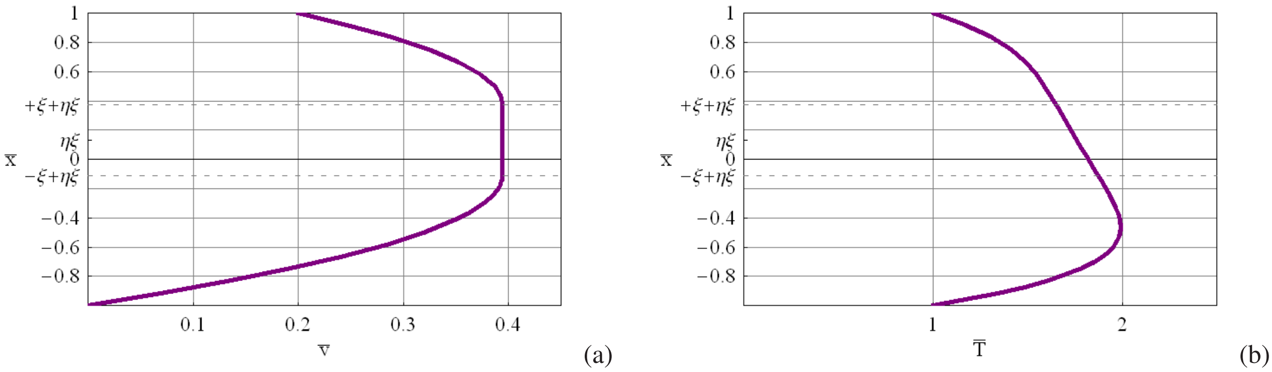

Fig. 1 (a) Velocity profile, (b) Induced temperature profile,

This is in second (polynomial) order in space. From the energy balance

$$-\kappa \frac{\mathrm{d}^2 T}{\mathrm{d}x_2^2} = \sigma_{21} \frac{\mathrm{d}v_1}{\mathrm{d}x_2}\,,\ \bar{T} = \frac{T}{T_0}\,,\ \bar{p} = \frac{|p'|R}{k}\,,\ \bar{\kappa} = \frac{\kappa T_0}{k v_0 R}\,, \tag{6}$$

one obtains the temperature profile[1] :

$$\bar{T} = \frac{\bar{p}}{12\bar{\kappa}}\Big(1 - \bar{x}^4 - 2(\bar{x}^3 \pm 1)\alpha + 6(\bar{x}^2+1)\beta\Big) + c(\bar{x} \mp 1) + 1\,, \tag{7}$$

$$\alpha = 2\eta\xi \pm 2\xi \mp \bar{k}\,,\ \beta = \xi^2(1+\eta^2) \pm 2\eta\xi^2 - \bar{k}\xi(1 \pm \eta)\,, \tag{8}$$

by assuming identical temperatures at the boundaries $T|_{\pm 1} = T_0$, identical heat fluxes in the transition points $\bar{\kappa}\frac{\mathrm{d}\bar{T}}{\mathrm{d}\bar{x}}|_{\xi\eta+\xi} = \bar{\kappa}\frac{\mathrm{d}\bar{T}}{\mathrm{d}\bar{x}}|_{\xi\eta-\xi} = \bar{\kappa}c$, and the same thermal conductivities in both regimes. The temperature profile is of fourth order in space. It leads to a production of entropy as derived in [2] and to a dissipation function Φ mentioned in [3] :

$$\Sigma = \frac{\kappa}{T^2}\left(\frac{\mathrm{d}T}{\mathrm{d}x_2}\right)^2 + \frac{1}{T}\sigma_{21}\frac{\mathrm{d}v_1}{\mathrm{d}x_2}\,,\ \bar{\Sigma} = \frac{T_0 R}{k v_0}\Sigma \tag{9}$$

$$\Phi = \bar{T}\bar{\Sigma} = \frac{\bar{\kappa}}{\bar{T}}\left(\frac{\mathrm{d}\bar{T}}{\mathrm{d}\bar{x}}\right)^2 + \frac{1}{\bar{k}}\left(\frac{\mathrm{d}\bar{v}}{\mathrm{d}\bar{x}}\right)^2 \mp \frac{\mathrm{d}v}{\mathrm{d}\bar{x}} \quad . \tag{10}$$

This function is assumed to be a measure of irreversibility. It contains one mechanical term (by using (5))

$$\Phi_{\mathrm{M}} = \frac{1}{\bar{k}}\left(\frac{\mathrm{d}\bar{v}}{\mathrm{d}\bar{x}}\right)^2 \mp \frac{\mathrm{d}v}{\mathrm{d}\bar{x}} = \frac{1}{\bar{k}}\Big(-\bar{x} \pm \xi + \eta\xi\Big)^2 \mp \Big(-\bar{x} \pm \xi + \eta\xi\Big)\,, \tag{11}$$

and one thermal term (by using (7))

$$\Phi_{\mathrm{T}} = \frac{\bar{\kappa}}{\bar{T}}\left(\frac{\mathrm{d}\bar{T}}{\mathrm{d}\bar{x}}\right)^2 = \frac{1}{\bar{T}}\left(\frac{\bar{p}}{\bar{\kappa}}\Big(\frac{\bar{x}^6}{9} - \frac{\bar{x}^5}{3}\alpha + \bar{x}^4(\frac{2}{3}\beta + \frac{\alpha^2}{4}) - \bar{x}^3\beta + \bar{x}^2\beta^2\Big) + \bar{\kappa}c^2 - 2c\bar{p}\Big(\frac{\bar{x}^3}{3} - \frac{\bar{x}^2}{2}\alpha + \bar{x}\beta\Big)\right) \quad . \tag{12}$$

Note that because of $\bar{T}$ both of them are of second order in space. We used a constitutive relation for stress depending only on the velocity field. We also chose identical temperatures at the boundaries, and thus suppressed additional temperature gradients in the steady state solution. The temperature gradients result only from the velocity gradients. The thermal and the mechanical parts of the dissipation function are both second order in space. If the dissipation function $\Phi = \Phi_{\mathrm{M}} + \Phi_{\mathrm{T}}$ is a measure of irreversibility, then the role of the temperature gradients should not be disregarded. Consequently, trying to express any irreversible process and neglecting the role of the temperature gradients, would be inappropriate.

References

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[1] notation $\pm$ refers to a $+$ in $\xi + \eta\xi \leq \bar{x} \leq 1$, $-$ in $-\xi + \eta\xi \geq \bar{x} \geq -1$ and vice-versa

