

Establishing Experiments to Find Material Functions in Tensor Nonlinear Constitutive Relations

D. V. Georgievskii^a, W. H. Müller^b, and B. E. Abali^b

^aFaculty of Mechanics and Mathematics, Moscow State University, Moscow, 119991 Russia

^bTechnische Universität Berlin, Charlottenburg, 10623 Berlin, Germany

e-mail: georgiev@mech.math.msu.su

Abstract—The basic scheme of an establishing experiment to find two material functions of two invariants in the constitutive relations of a nonlinear incompressible tensor medium, is described. A combination of radial spreading and two one-dimensional shears in mutually perpendicular directions observed in a cylindrical layer is taken as the base flow of such a medium. Percolation is possible at the boundaries of cylinders, but the so-called tangential component of the medium coincides with the velocity of cylinder motion. The fundamental question of a viscous potential existing for each medium is posed.

DOI: 10.3103/S1062873812120131

NONLINEAR ISOTROPIC TENSOR FUNCTIONS AND THEIR INVARIANTS: REINER–RIVLIN MEDIA

It is known from tensor analysis [1–3] that the most common form of the polynomial isotropic tensor function of one argument that associates two symmetric second-rank tensors $s(\vec{x}, t)$ and $d(\vec{x}, t)$ with zero traces ($\text{tr} \underline{s} \equiv I_{s1} = 0$, $\text{tr} \underline{d} \equiv I_{d1} = 0$), i.e., two deviators, is

$$s = A_1(I_{d2}, I_{d3})\underline{d} + A_2(I_{d2}, I_{d3})(\underline{d}^2 - I_{d2}\underline{I}/3), \quad (1)$$

$$\underline{d} = B_1(I_{s2}, I_{s3})\underline{s} + B_2(I_{s2}, I_{s3})(\underline{s}^2 - I_{s2}\underline{I}/3), \quad (2)$$

$$I_{d2} = \sqrt{\text{tr} \underline{d}^2}, \quad I_{d3} = \sqrt[3]{\text{tr} \underline{d}^3}, \quad (3)$$

$$I_{s2} = \sqrt{\text{tr} \underline{s}^2}, \quad I_{s3} = \sqrt[3]{\text{tr} \underline{s}^3},$$

where \underline{I} is a second-rank unit tensor, and I_{d2} , I_{s2} , I_{d3} , and I_{s3} are quadratic and cubic invariants. The first two serve as measures of tensor \underline{d} and \underline{s} in R^3 .

$$I_{d2} = \|\underline{d}\|, \quad I_{s2} = \|\underline{s}\|; \quad I_{d2} = 0 \Leftrightarrow \underline{d} \equiv \underline{0},$$

$$I_{s2} = 0 \Leftrightarrow \underline{s} \equiv \underline{0}.$$

Let us assume that objects \underline{s} and \underline{d} have the physical sense of deviators of stress tensor $\underline{\sigma}(\vec{x}, t) = -p\underline{I} + \underline{s}$ (where $p = -\text{tr} \underline{\sigma}/3$ is pressure) and the strain-rate tensor at each point $\vec{x} \in R^3$ of an incompressible medium. The incompressibility tells us that deviator \underline{d} coincides with the strain-rate tensor itself. Relations (1) and (2) are then mutually reversible and tensor nonlinear, and determine the material functions A_1 , A_2 , B_1 , and B_2 of the quadratic and cubic invariants of corre-

sponding tensors. Specifying one pair of the functions $\{A_1, A_2\}$ or $\{B_1, B_2\}$ determines the properties of a medium in the class of determining relationships (1) and (2). The physical dimensionalities of the introduced material functions are

$$[A_1] = 1/[B_1] = ML^{-1}T^{-1}, [A_2] = ML^{-1}, [B_2] = M^{-2}L^2T^3.$$

Relationships (1) and (2), along with the Hamilton–Kelly theorem, which allows us to express the traces of higher tensor ranks through their quadratic and cubic invariants, make it possible to associate I_{s2} , I_{s3} with I_{d2} , I_{d3} :

$$I_{s2}^2 = I_{d2}^2 A_1^2 + 2I_{d3}^3 A_1 A_2 + \frac{1}{6} I_{d2}^4 A_2^2, \quad (4)$$

$$I_{s3}^3 = I_{d3}^3 A_1^3 + \frac{1}{2} I_{d2}^4 A_1^2 A_2 \quad (5)$$

$$+ \frac{1}{2} I_{d2}^2 I_{d3}^3 A_1 A_2^2 - \frac{1}{36} I_{d2}^6 A_2^3 + \frac{1}{3} I_{d3}^6 A_2^3,$$

i.e., to derive the scalar constitutive relations of the medium.

We also note the expressions of material functions B_1 , B_2 through A_1 , A_2 :

$$B_1 = \frac{-6A_1^2 + I_{d2}^2 A_2^2}{-6A_1^3 + 3I_{d2}^2 A_1 A_2^2 + 2I_{d3}^3 A_2^3}, \quad (6)$$

$$B_2 = \frac{6A_2}{-6A_1^3 + 3I_{d2}^2 A_1 A_2^2 + 2I_{d3}^3 A_2^3}.$$

In the continuum mechanics (and particularly in hydrodynamics), the nonlinear tensor fluids described by relationships (1) and (2) are referred to as Reiner–Rivlin media. Their properties have been investigated in numerous works (see reviews in [4–7]). The

Reiner–Rivlin models are involved in investigations if the unity direction tensors of stresses $\underline{s}^0 = \underline{s}/\|\underline{s}\|$ and strain-rates $\underline{d}^0 = \underline{d}/\|\underline{d}\|$ differ substantially, and we cannot ignore this difference.

If $\underline{s}^0 = \underline{d}^0$, there is no quadratic tensor nonlinearity in (1), (2), and $A_2 \equiv 0$, $B_2 \equiv 0$, testifying to the tensor nonlinearity or quasi-nonlinearity of function $\underline{s}(\underline{d})$.

Among quasi-linear media, a Newtonian viscous fluid is the only one that is physically linear: $A_1 = 2\mu$ and $B_1 = 1/(2\mu)$, where μ is dynamic viscosity independent of I_{d2} and I_{d3} . The physical linearity of operator $\underline{s}(\underline{d})$ is equivalent to fulfilling the superposition principle, making it considerably easier to solve the problem.

COMBINING SPREAD AND ONE-DIMENSIONAL SHEARS IN TWO DIRECTIONS

Let us consider the steady flow of a medium with constitutive relations (1) in a cylindrical layer, in a cylindrical set of coordinates (r, θ, z) :

$$\Omega = \{R_1 < r < R_2, 0 \leq \theta < 2\pi, -\infty < z < \infty\}, \quad (7)$$

$$R_1, R_2 - \text{const.}$$

Let us specify velocity field \vec{v} in Ω by the following components:

$$\begin{aligned} v_r &= \frac{Q}{r}, \quad v_\theta = v_\theta(r), \\ v_z &= v_z(r), \quad 0 < Q - \text{const.} \end{aligned} \quad (8)$$

so that the incompressibility condition is fulfilled identically. Let us write the nonzero components of deformation rates and invariants I_{d2} and I_{d3} :

$$d_{rr} = -d_{\theta\theta} = -\frac{Q}{r}, \quad d_{r\theta} = \frac{1}{2}\left(v'_\theta - \frac{v_\theta}{r}\right), \quad d_{rz} = \frac{v'_z}{2}, \quad (9)$$

$$I_{d2} = \sqrt{2\left(\frac{Q^2}{r^4} + d_{r\theta}^2 + d_{rz}^2\right)}, \quad I_{d3} = -\sqrt[3]{\frac{3Qd_{rz}^2}{r^2}} \quad (10)$$

According to (1), kinematics (9) causes stresses $s(r)$ in the medium:

$$\begin{aligned} s_{rr} &= -A_1 \frac{Q}{r^2} + \frac{A_2}{3} \left(\frac{Q^2}{r^4} + d_{r\theta}^2 + d_{rz}^2 \right), \\ s_{\theta\theta} &= A_1 \frac{Q}{r^2} + \frac{A_2}{3} \left(\frac{Q^2}{r^4} + d_{r\theta}^2 - 2d_{rz}^2 \right), \\ s_{zz} &= -\frac{A_2}{3} \left(\frac{2Q^2}{r^4} + 2d_{r\theta}^2 - d_{rz}^2 \right), \\ s_{r\theta} &= A_1 d_{r\theta}, s_{rz} = \left(A_1 - A_2 \frac{Q}{r^2} \right) d_{rz}, s_{\theta z} = A_2 d_{r\theta} d_{rz}. \end{aligned} \quad (11)$$

On the other hand, deviator of stresses $\underline{s}(r)$, pressure $p(r)$, and velocity $\vec{v}(r)$ are connected by three equations of motion (mass forces \vec{F} have components $F_r = F_\theta = 0$, $F_z = -g$):

$$\begin{aligned} (-p + s_{rr})' + \frac{1}{r}(s_{rr} - s_{\theta\theta}) - \rho \left(\frac{Q^2}{r^3} + \frac{v_\theta^2}{r} \right), \\ s'_{r\theta} + \frac{2s_{r\theta}}{r} = \rho Q \left(\frac{v'_\theta}{r} + \frac{v_\theta}{r^2} \right), \\ s'_{rz} + \frac{s_{rz}}{r} = \rho Q \frac{v'_z}{r} + \rho g. \end{aligned} \quad (12)$$

The first Eq. (12) serves to determine the pressure, and the second and third introduce two first integrals:

$$\begin{aligned} s_{r\theta} = \frac{\rho Q}{r} \left(v_\theta + \frac{\alpha}{r} \right), \quad s_{rz} = \frac{\rho Q}{r} (v_z + \beta) + \frac{\rho g r}{2}, \\ \alpha, \beta - \text{const.} \end{aligned} \quad (13)$$

Substituting constitutive relations (11) into (13), we derive the nonlinear set of second-order differential equations relative to $v_\theta(r)$ and $v_z(r)$:

$$\begin{aligned} A_1(I_{d2}, I_{d3}) \left(v'_\theta - \frac{v_\theta}{r} \right) &= \frac{2\rho Q}{r} \left(v_\theta + \frac{\alpha}{r} \right), \\ \left(A_1(I_{d2}, I_{d3}) - \frac{Q}{r^2} A_2(I_{d2}, I_{d3}) \right) v'_z &= \\ = \frac{2\rho Q}{r} (v_z + \beta) + \rho g r. \end{aligned} \quad (14)$$

$$(15)$$

Specifying the material functions A_1 and A_2 of invariants (10) makes it possible in principle to integrate set (14), (15), i.e., to find velocities $v_\theta(r)$ and $v_z(r)$, the expressions for which, along with α and β , involve two additional constants. To find all four constants, we use the four kinematic boundary conditions

$$v_\theta|_{r=R_1} = V_\theta, \quad v_z|_{r=R_1} = V_z, \quad v_\theta|_{r=R_2} = v_z|_{r=R_2} = 0. \quad (16)$$

Note that as follows from (8), there is no adhesion at cylindrical boundaries $r = R_1$ and $r = R_2$, since the medium streams through them:

$$v_r|_{r=R_1} = Q/R_1, \quad v_r|_{r=R_2} = Q/R_2. \quad (17)$$

PROBLEM OF IDENTIFYING THE MATERIAL FUNCTIONS

We have described the basic schematic for solving the direct problem of finding components $v_\theta(r)$ and $v_z(r)$ from the known material functions A_1 and A_2 of constitutive relations (1). Let us now use it to consider the inverse problem, which traditionally plays a large

role in the experimental theory of constitutive relations and experimental hydrodynamics of non-Newtonian media [8–13]: the problem of identifying the properties of a tensor nonlinear medium and the effects of low orders of infinitesimality. To investigate it, we propose the following establishing experiment.

Assume that vertical cylindrical surface $r = R_1$ rotates around its axis with angular velocity V_θ/R_1 in a gravitational force field with acceleration g , moves along the generatrix with velocity V_z , and emits the investigated medium with normal velocity Q/R_1 . The volume of medium emitted over time dt in amount dz is thus $2\pi Q dz dt$. Another cylindrical surface, $r = R_2 > R_1$, is immobile and can absorb the incoming medium. By virtue of incompressibility, the volume of the medium absorbed by the outer surface over time dt in segment dz naturally equals $2\pi Q dz dt$ as well.

A steady flow is observed in region Ω (7). It is a combination of spreading and two one-dimensional shears. Tangential adherence occurs on cylinders $r = R_1$ and $r = R_2$; i.e., the tangential component $\pm \sqrt{|\vec{v}|^2 - v_r^2}$ of the velocity of each particle located at the boundary coincides with the boundary velocity at this point. Flow rate $Q > 0$ is fixed, while parameter V_θ and V_z can be varied in an experiment.

Let us solve set (14), (15) relative to material functions A_1 and A_2 :

$$\begin{aligned} A_1 &= \frac{2\rho Q(rv_\theta + \alpha)}{r(rv'_\theta - v_\theta)}, \\ A_2 &= 2\rho r \left(\frac{rv_\theta + \alpha}{rv'_\theta - v_\theta} - \frac{v_z + \beta}{v'_z} - \frac{gr^2}{2Qv'_z} \right). \end{aligned} \quad (18)$$

We assume that from our measurements we can find with sufficient accuracy

(i) both profiles $v_\theta(r)$ and $v_z(r)$ (and thus calculate components $d_{r\theta}(r)$, $d_{rz}(r)$ of the strain-rates and invariants $I_{d2}(r)$, $I_{d3}(r)$ using (9) and (10));

(ii) tangential stresses $s_{r\theta}$ and s_{rz} at one of the boundaries, e.g., at immobile boundary $r = R_2$ (and thus calculate constants α and β using (13)).

The right sides in (18) become known. However, material functions (18) must depend on r through invariants $I_{d2}(r)$ and $I_{d3}(r)$. Since the latter are known functions of the radius, we can derive the relation $I_{d3} = I_{d3}(I_{d2})$. Its parametric version takes the form

$$I_{d2} = I_{d2}(r), \quad I_{d3} = I_{d3}(r), \quad R_1 < r < R_2. \quad (19)$$

Each flow of the investigated medium is thus reflected in a half-plane (I_{d2}, I_{d3}) at fixed Q , V_θ , and V_z

(by definition, $I_{d2} \geq 0$) of an open curve. Performing a set of experiments for various velocities (V_θ, V_z) with these curves covers the regions in half-plane (I_{d2}, I_{d3}) in which functions A_1 and A_2 become known.

VISCOUS POTENTIAL

Tensor function (1) has the scalar potential $W(I_{d1}, I_{d2}, I_{d3})|_{I_{d1}=0}$:

$$\begin{aligned} \underline{s} &= \frac{\partial W}{\partial \underline{d}} = \sum_{m=1}^3 \frac{\partial W}{\partial I_{dm}} \frac{\partial I_{dm}}{\partial \underline{d}} = \frac{\partial W}{\partial I_{d1}} \underline{I} + \frac{1}{I_{d2}} \frac{\partial W}{\partial I_{d2}} \underline{d} \\ &\quad + \frac{1}{I_{d3}^2} \frac{\partial W}{\partial I_{d3}} \underline{d}^2, \end{aligned} \quad (20)$$

$$\frac{\partial W}{\partial I_{d1}} = -\frac{1}{3} I_{d2}^2 A_2, \quad \frac{\partial W}{\partial I_{d2}} = I_{d2} A_1, \quad \frac{\partial W}{\partial I_{d3}} = I_{d3}^2 A_2, \quad (21)$$

if material functions A_1 and A_2 satisfy three conditions of potentiality that are equivalent to three equalities of mixed derivatives: $\partial^2 W / (\partial I_{dm} \partial I_{dn}) = \partial^2 W / (\partial I_{dn} \partial I_{dm})$. Since all our previous reasoning was for hyperplane $I_{d1} = 0$, then one of these three conditions for incompressible materials is of interest:

$$\frac{\partial^2 W}{\partial I_{d2} \partial I_{d3}} = \frac{\partial^2 W}{\partial I_{d3} \partial I_{d2}} \Rightarrow I_{d2} \frac{\partial A_1}{\partial I_{d3}} = I_{d3}^2 \frac{\partial A_2}{\partial I_{d2}}. \quad (22)$$

Are relations (22) fulfilled? If A_1 and A_2 are constants, then relation (22) is evident. For quasi-linear media ($A_2 \equiv 0$), it shows that A_1 does not depend on the third variant. Like the existence of viscous potential W , however, this does not follow from any continuum mechanics postulate. Finding material functions A_1 and A_2 in the above initial experiment will, with the subsequent verification of (22), allow us to answer this important theoretical question for each specific tensor nonlinear medium.

ACKNOWLEDGMENTS

This study was supported by the Russian Foundation for Basic Research, projects nos. 12-01-00020a, 11-01-00181a, and 12-01-00789a.

REFERENCES

1. Rivlin, R.S. and Ericksen, J.L., *J. Rational Mech. Anal.*, 1955, vol. 4, no. 2, p. 323.
2. Pobedrya, B.E., *Lektsii po tenzornomu analizu* (Lectures on Tensor Analysis), Moscow: Izd. MGU, 1986.
3. Georgievskii, D.V., *Usp. Mekh.*, 2002, vol. 1, no. 2, p. 150.

4. Astarita, G. and Marrucci, G., *Principles of Non-Newtonian Fluid Mechanics*, London: McGraw Hill, 1974; Moscow: Mir, 1978.
5. Skul'skii, O.I. and Aristov, S.N., *Mekhanika anomal'no vyazkikh zhidkosti* (Mechanics of Anomalously Viscous Fluids), Moscow: Izd. RKhD, 2004.
6. Georgievskii, D.V., *Nelinein. Dinam.*, 2011, vol. 7, no. 3, p. 451.
7. Müller, W.H. and Abali, B.E., *RAMM*, 2012, vol. 12, no. 1.
8. Weissenberg, K., *Nature*, 1947, vol. 159, no. 4035, p. 310.
9. McKennell, R., *Anal. Chem.*, 1956, vol. 28, no. 11, p. 1710.
10. Beavers, G.S. and Joseph, D.D., *J. Fluid Mech.*, 1975, vol. 69, pt. 3, p. 475.
11. Steffe, J.F., *Rheological Methods in Food Process Engineering*, Michigan: Freeman Press, 1996.
12. Ewoldt, R.H., Hosoi, A.E., and McKinley, G.H., *J. Rheolog.*, 2008, vol. 52, no. 6, p. 1427.
13. Georgievskii, D.V., *Bull. Russ. Acad. Sci. Phys.*, 2011, vol. 75, no. 1, p. 140.