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Inverse analysis of a non-linear viscous fluid based on dissipated energy measured with a simple-shear rheometer

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Oscillatory rheometer measurements are used to determine the material parameters of a NEWTONian fluid model, which can be expressed by a linear constitutive relation. However, rheological materials, such as polymer melts, mixture of oils, or food paste, can only be modeled as non-NEWTONian fluids by using non-linear constitutive relations. Since the rheometer measures the energy loss in the induction motor due to shear loading of the viscous material, this can be used as the objective function for a regression analysis. The dissipated energy will be obtained as outlined in [1]. The goal of this work is to explain how to determine the parameters of a non-linear material model by using the energy loss that is measured in a rheometer.

1 Energy loss in a shear flow of a non-linear stress response

Consider a plate-plate (or cone-plate) rheometer such that the cylindrical (or conical) sample, Ω, is sheared under rotational movement of the upper plate (or cone). This can be modeled as a two-dimensional shear flow in the circumferential to height directions. In a motion controlled simple-shear rotary viscometer, the induced velocity gradients in such a shear flow reads

$$d_{ij} = \frac{\partial v_{(i}}{\partial x_{j)}} = \frac{1}{2}\Big(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\Big) = \begin{pmatrix} 0 & d \\ d & 0 \end{pmatrix} , \tag{1}$$

where the motion is periodic with a frequency ν. By varying the frequency we can measure the rate-dependency. When we use a polynomial expansion for the generic representation of the symmetric stress tensor as outlined in [2] and construct a third-order fluid with the invariants of the velocity gradients $I_{\boldsymbol{d}}, II_{\boldsymbol{d}}, III_{\boldsymbol{d}}$, such as

$$\begin{aligned}\sigma_{ij} = \big(c_1 + c_2 I_{\boldsymbol{d}} + c_3 II_{\boldsymbol{d}} + c_4 I_{\boldsymbol{d}}^2 + c_5 III_{\boldsymbol{d}} + c_6 I_{\boldsymbol{d}} II_{\boldsymbol{d}}\big)\delta_{ij} + \big(c_7 + c_8 I_{\boldsymbol{d}} + c_9 II_{\boldsymbol{d}} + c_{10} I_{\boldsymbol{d}}^2\big) d_{ij} + \\ + \big(c_{11} + c_{12} I_{\boldsymbol{d}}\big) d_{ik} d_{kj} \ , I_{\boldsymbol{d}} = d_{ii}, II_{\boldsymbol{d}} = d_{ij} d_{ji}, III_{\boldsymbol{d}} = d_{ij} d_{jk} d_{ki} \ ,\end{aligned} \tag{2}$$

then in one cycle the energy loss from an isothermal system, i.e., $\partial T/\partial x_i \equiv 0$, reads (cf. [1]):

$$\begin{aligned}E_{\text{loss}} = \sum_{t \in per} (t - t^0) \int_\Omega \Phi \, \mathrm{d}v \, , \ \Phi = \Sigma T \, , \ \Sigma = \frac{\kappa}{T^2} \frac{\partial T}{\partial x_i} \frac{\partial T}{\partial x_i} + \frac{1}{T} \sigma_{ij} d_{ji} = \frac{1}{T} \Big(\big(c_1 + c_2 I_{\boldsymbol{d}} + c_3 II_{\boldsymbol{d}} + c_4 I_{\boldsymbol{d}}^2 + \\ + c_5 III_{\boldsymbol{d}} + c_6 I_{\boldsymbol{d}} II_{\boldsymbol{d}}\big) d_{ii} + \big(c_7 + c_8 I_{\boldsymbol{d}} + c_9 II_{\boldsymbol{d}} + c_{10} I_{\boldsymbol{d}}^2\big) d_{ij} d_{ji} + \big(c_{11} + c_{12} I_{\boldsymbol{d}}\big) d_{ik} d_{kj} d_{ji} \Big) \ .\end{aligned} \tag{3}$$

We may also use the invariants and rewrite the dissipation function: $\Phi = c_1 I_{\boldsymbol{d}} + c_2 I_{\boldsymbol{d}}^2 + c_3 I_{\boldsymbol{d}} II_{\boldsymbol{d}} + c_4 I_{\boldsymbol{d}}^3 + c_5 I_{\boldsymbol{d}} III_{\boldsymbol{d}} + c_6 I_{\boldsymbol{d}}^2 II_{\boldsymbol{d}} + c_7 II_{\boldsymbol{d}} + c_8 I_{\boldsymbol{d}} II_{\boldsymbol{d}} + c_9 II_{\boldsymbol{d}}^2 + c_{10} I_{\boldsymbol{d}}^2 II_{\boldsymbol{d}} + c_{11} III_{\boldsymbol{d}} + c_{12} I_{\boldsymbol{d}} III_{\boldsymbol{d}}$.

2 Regression based on the energy loss

The so-called loss module, G'', is determined in association with the linear theory where a sine strain function, $\varepsilon = \varepsilon_0 \sin(\omega t)$, creates a sine response function with a phase shift, $\sigma = B \sin(\omega t + \delta)$. Then the viscoelastic response, $\sigma = G' \varepsilon + \frac{1}{\omega} G'' \varepsilon^\bullet$, enables us to determine the energy loss for the process with homogeneous shear strain rate and stress

$$\begin{aligned}E_{\text{loss}} = \int_0^{1/\nu} \int_\Omega \sigma \varepsilon^\bullet \, \mathrm{d}v \, \mathrm{d}t = V \int_0^{1/\nu} \sigma \varepsilon^\bullet \, \mathrm{d}t = V \int_0^{1/\nu} \Big(G' \varepsilon \varepsilon^\bullet + \frac{G''}{\omega} (\varepsilon^\bullet)^2 \Big) \, \mathrm{d}t = \\ = V \int_0^{2\pi/\omega} \Big(G' \varepsilon_0^2 \omega \sin(\omega t) \cos(\omega t) + G'' \varepsilon_0^2 \omega \cos^2(\omega t) \Big) \, \mathrm{d}t = V G'' \varepsilon_0^2 \omega \frac{1}{2} \frac{2\pi}{\omega} = V G'' \varepsilon_0^2 \pi \ .\end{aligned} \tag{4}$$

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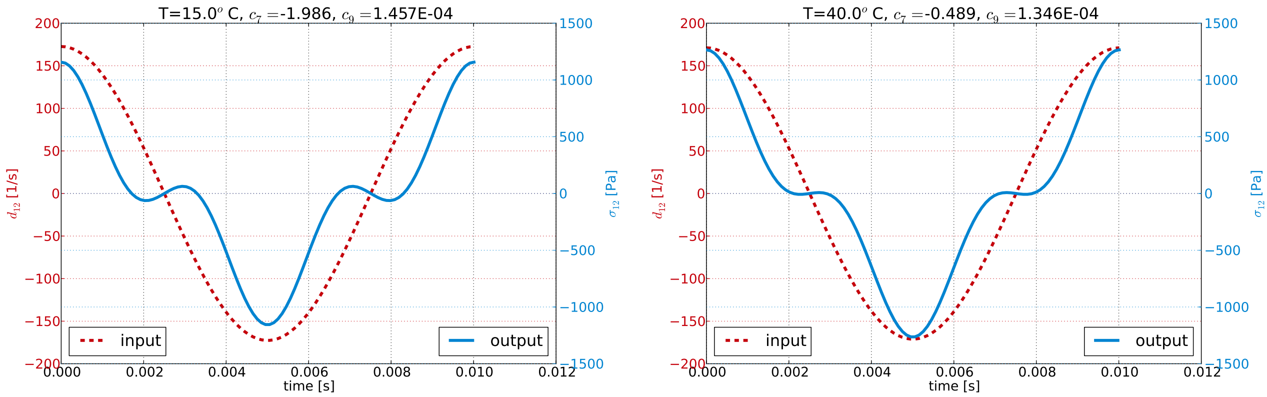

Fig. 1: Stress vs. velocity gradient of a non-linear oil determined from the rheometer experiments.

For a purely viscous fluid without elasticity G' vanishes and an experiment in a rheometer provides ε_0, G''.

For the case described by Eq. (1) the invariants become $I_{\boldsymbol{d}} = 0, II_{\boldsymbol{d}} = 2d^2, III_{\boldsymbol{d}} = 0$ and therefore the dissipation function is $\Phi = c_7 II_{\boldsymbol{d}} + c_9 II_{\boldsymbol{d}}^2$. Hence the energy loss for a set of parameters $\{c_7, c_9\}$ stems from the Eq. (3) under the assumption of a homogeneous velocity gradient in the probe, such as $E_{\text{loss}} = \sum_{t \in per}(t - t^0)V(c_7 II_{\boldsymbol{d}} + c_9 II_{\boldsymbol{d}}^2)$. We shall perform experiments at different frequencies and calculate the energy loss with Eq. (4). Theoretically only two different frequencies would be sufficient in order to determine the two parameters $\{c_7, c_9\}$. However, in a range of frequencies we obtain more than two results and, hence, the inverse problem is ill-conditioned. As known from linear regression, for m-different frequencies $\{\nu^1, \nu^2, \dots, \nu^m\}$ the set of equations

$$\begin{pmatrix} \sum_{t \in per}(t - t^0) II_{\boldsymbol{d}}|_{\nu^1} & \sum_{t \in per}(t - t^0) II_{\boldsymbol{d}}^2|_{\nu^1} \\ \sum_{t \in per}(t - t^0) II_{\boldsymbol{d}}|_{\nu^2} & \sum_{t \in per}(t - t^0) II_{\boldsymbol{d}}^2|_{\nu^2} \\ \vdots & \vdots \\ \sum_{t \in per}(t - t^0) II_{\boldsymbol{d}}|_{\nu^m} & \sum_{t \in per}(t - t^0) II_{\boldsymbol{d}}^2|_{\nu^m} \end{pmatrix} \begin{pmatrix} c_7 \\ c_9 \end{pmatrix} = \begin{pmatrix} G''|_{\nu^1} \varepsilon_0^2 \pi \\ G''|_{\nu^2} \varepsilon_0^2 \pi \\ \vdots \\ G''|_{\nu^1} \varepsilon_0^2 \pi \end{pmatrix} \Rightarrow A_{ij} u_j = b_i \tag{5}$$

can be solved by using the “normal equation" such as $u_k = \big(A_{ij} A_{ik}\big)^{-1} A_{ij} b_i$.

We have used a cone-plate simple-shear rheometer AR1000 of TA Instruments$^{\text{TM}}$ with a cone of 60 mm diameter and 2° cone angle. The material is a mixture of natural essential oils from Weleda$^{\text{TM}}$. We have performed oscillatory tests for two temperatures by controlling the strain with a sine function of amplitude 0.3 and by varying the frequency between $1 - 100$ Hz. The results have been used in order to determine the parameters c_7, c_9 in Eq. (3). The results can be seen in Fig. 1. The velocity gradient is the input function, since strain is controlled, and the shape of it has been changed due to the non-linear character of the fluid. The assumption that the used mixture of oils is purely viscous may be incorrect and a more detailed study regarding viscoelastic fluids has been left for further studies.

References

[1] W. H. Müller and B. E. Abali, PAMM **12**(1) (2012).
[2] A. J. M. Spencer, in: Theory of invariants, edited by A. C. Eringen (Academic Press Inc. London, 1971), chap. Part III, pp. 239–352.