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Generalizations of the Orr–Sommerfeld Problem for the Case in Which the Unperturbed Shear Motion is Nonsteady

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To Viktor Antonovich Sadovnichii on the occasion of his 75th birthday

Abstract. We consider problems of the linearized theory of hydrodynamic stability for the case in which the unperturbed plane-parallel-flow of a viscous incompressible fluid in a layer is substantially unsteady. We analyze the Orr–Sommerfeld equation, which is generalized for this case, with different combinations of the four boundary conditions specified on the straight parts of the boundaries of the layer. Using the apparatus of integral relations, including, in particular, the analysis of the minimization problem for quadratic functionals, we derive upper bounds for the growth or decay of kinematic perturbations with respect to the integral measure. A special attention is paid to the longitudinal oscillation mode of the layer, to the power-law acceleration or deceleration, and also to the process similar to the diffusion of the vortex layer. An investigation of the reducibility of the three-dimensional picture of perturbations imposed on a plane-parallel unsteady shift to a two-dimensional picture in the plane of this shift is carried out. Generalizations of the Squire theorem are established.

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1. LINEARIZED STABILITY EQUATION OF UNSTEADY SHEAR

The Orr–Sommerfeld equation, which is well known in mathematical physics and in the linearized theory of hydrodynamic stability [1–5],

$$\frac{1}{\mathrm{Re}}(\varphi^{IV} - 2s^2\varphi'' + s^4\varphi) = (\alpha + isv^\circ)(\varphi'' - s^2\varphi) - isv^{\circ\prime\prime}\varphi, \quad 0 < x < 1, \tag{1.1}$$

describes the time evolution of perturbations imposed on a steady one-dimensional shear flow of an incompressible Newtonian fluid in a plane layer $\Omega = \{-\infty < x_1 < \infty, 0 < x_2 < 1\}$ characterized by the profile of the longitudinal velocity $v_1^\circ(x_2) \equiv v^\circ(x)$, $v_2^\circ \equiv 0$, $x \equiv x_2$. The unknown complex-valued function $\varphi(x)$ is related to a real perturbation of the stream function ψ ($v_1 = \partial\psi/\partial x_2$, $v_2 = -\partial\psi/\partial x_1$) as follows

$$\psi(x_1, x, t) = \varphi(x)\mathrm{e}^{isx_1+\alpha t}, \tag{1.2}$$

where $s > 0$ is the wave number of a selected harmonic along the x_1 axis; the spectral parameter $\alpha = \alpha_* + i\alpha_{**}$ is the complex frequency such that the stability as $t \to \infty$ (in the linearized approximation, or in the small) of the separate harmonic with the wave number s is related to the sign of the real part $\alpha_*(s)$ of this frequency. If all branches $\alpha_{*j}(s)$ $(j = 1, 2, \dots)$ for every $s > 0$ belong to the left half-plane of the complex plane α, then the basic flow with the profile $v^\circ(x)$ is stable with respect to small perturbations.

The Orr–Sommerfeld equation (1.1) is written in a dimensionless form. The dimensional basis includes the thickness h of the layer, the characteristic velocity V, and the fluid density ρ; $\mathrm{Re} = \rho V h/\mu$ stands for the Reynolds number and μ for the dynamic viscosity.

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Along with the plane-parallel velocity field, we shall characterize the basic motion of the stress tensor with the components

$$\sigma_{11}^{\circ} = \sigma_{22}^{\circ} = -p^{\circ}, \quad \sigma^{\circ} \equiv \sigma_{12}^{\circ} = \frac{v^{\circ\prime}}{\mathrm{Re}}, \tag{1.3}$$

where p° stands for the pressure. Here and below, the parameters of the nonperturbed viscous flow are marked by the superscript $\circ$; the signs of variations before the perturbations are omitted for brevity; the primes stand for the derivatives with respect to the variable x of the quantities depending on x only.

Equation (1.1), together with the family of four linear homogeneous boundary conditions (two of them are posed on each of the boundaries $x = 0$ and $x = 1$), forms an eigenvalue problem of order four with spectral parameter α. It is reasonable to consider quadruples of boundary conditions that have definite mechanical meaning. Clearly, among them, the simplest family from the viewpoint of mathematical formulation is

$$\varphi(0) = \varphi(1) = 0, \quad \varphi'(0) = \varphi'(1) = 0, \tag{1.4}$$

which corresponds to the adhesion of the medium to both the boundaries that remain straight and parallel to each other in the perturbed motion. The spectral problem (1.1), (1.4) is referred to as the *Orr–Sommerfeld problem*, and it has a centennial history of research, starting with [6, 7].

Other possible conditions at one of the boundaries of the layer (e.g., $x = 1$) can be

— prescribing the shear stress $\sigma^{\circ}(t)$ on the boundary $x = 1$, which remains straight in the perturbed motion,

$$\varphi(1) = 0, \quad \varphi''(1) = 0, \tag{1.5}$$

— the condition of free boundary which becomes unknown and curvilinear in the perturbed motion [8],

$$x = 1: \quad \alpha(\varphi'' + s^2\varphi) = isv^{\circ\prime\prime}\varphi, \quad \frac{\alpha}{\mathrm{Re}}(\varphi'' - 3s^2\varphi)' = \alpha(\alpha + isv^{\circ})\varphi' - s^2 p^{\circ\prime}\varphi, \tag{1.6}$$

— the conditions that are a certain approximation of the previous ones [9],

$$x = 1: \quad \varphi' + s\varphi = 0, \quad \varphi''' + s\varphi'' = 0. \tag{1.7}$$

What generalizations to the setting of the linearized stability problem of a plane-parallel flow of a viscous fluid can be introduced by the fact that the kinematic and force factors v°, σ°, and p° become explicitly depending in time? We are talking about the stages of acceleration and deceleration of the flow (for example, under the influence of a changing pressure drop), about the vibrational modes, about the diffusion of the vortex sheet, and other types of basic motion such that the unsteadiness is of importance in the investigation of motion and cannot be neglected. Here the picture of nonperturbed flow still does not depend on the coordinate x_1.

The functions $v^{\circ}(x,t)$ and $\sigma^{\circ}(x,t)$ are the well-defined exact solutions of some initial-boundary problems for the Navier–Stokes system that define the basic flow of the fluid. However, in the literature on mathematical hydrodynamics, for the generality of the presentation, it is conventionally assumed that the profiles of the longitudinal velocity v° and the shear stress σ° are arbitrary functions that are twice and once continuously differentiable with respect to x, respectively.

If the known coefficients of the stability equation definitely depend on t, then it is impossible to single out (like (1.2)) the factor $\mathrm{e}^{\alpha t}$ in the perturbation of the stream function $\psi(x_1, x, t)$, and hence to form a spectral parameter (complex frequency) α which is completely "responsible" for the stability. Thus, in general, the linearized stability problem of unsteady shear cannot be reduced to investigating the dependence of the eigenvalues on the external parameters, such as the Reynolds number.

Carrying out the calculations (see, e.g., [10]) similar to the development of the very Orr–Sommerfeld equation (1.1) with respect to the function $\varphi(x,t)$ such that

$$\psi(x_1, x, t) = \varphi(x,t)\mathrm{e}^{isx_1}, \tag{1.8}$$

we obtain the following equation:

$$\frac{1}{\mathrm{Re}}(\varphi_{xxxx} - 2s^2\varphi_{xx} + s^4\varphi) = \Big(\frac{\partial}{\partial t} + isv^\circ\Big)(\varphi_{xx} - s^2\varphi) - isv^\circ_{xx}\varphi, \tag{1.9}$$

$$0 < x < 1, \quad t > 0,$$

generalizing (1.1), because in the special case of steady basic motion, the operator $\partial/\partial t$ is equivalent by (1.2) to the multiplication by the spectral parameter α. The subscripts x and t in (1.9) mean the partial differentiation with respect to the corresponding variables.

2. APPLICATION OF THE METHOD OF INTEGRAL RELATIONS

Suppose that the complex-valued function $\varphi(x,t)$ is an element of the Hilbert space $H_2(0;1)$, $t > 0$, and

$$\|\varphi\|^2(t) = \int_0^1 |\varphi|^2(x)\,dx < \infty. \tag{2.1}$$

Multiply equation (1.9) by $\bar\varphi$ and integrate with respect to x from 0 to 1. In terms of the quadratic functionals I_n and J depending on t, write

$$\frac{1}{\mathrm{Re}}\Big[I_2^2 + 2s^2I_1^2 + s^4I_0^2 + \big(\varphi_{xxx}\bar\varphi - \varphi_{xx}\bar\varphi_x - 2s^2\varphi_x\bar\varphi\big)_{x=0}^{x=1}\Big]$$
$$= -\frac{1}{2}\frac{d}{dt}(I_1^2 + s^2I_0^2) + [(\varphi_{xt} + isv^\circ\varphi_x)\bar\varphi]_{x=0}^{x=1} - is\int_0^1 v^\circ_x\varphi_x\bar\varphi\,dx + iJ, \tag{2.2}$$

$$I_n^2(t) = \int_0^1 \left|\frac{\partial^n\varphi}{\partial x^n}\right|^2 dx \equiv \left\|\frac{\partial^n\varphi}{\partial x^n}\right\|^2, \qquad n = 0,1,2, \tag{2.3}$$

$$J = -s\int_0^1 \Big(v^\circ_{xx}|\varphi|^2 + v^\circ\big(|\varphi_x|^2 + s^2|\varphi|^2\big)\Big)dx - \int_0^1 (\varphi_{xt}\bar\varphi_x + s^2\varphi_t\bar\varphi)_{**}\,dx \in R. \tag{2.4}$$

The off-integral summands in both the parts of (2.2) depend on the boundary conditions imposed on φ.

The objective of the apparatus of integral relations [11–13], which is intensively developed as applied to linearized stability problems for flows, is the derivation of an upper bound for the growth of decay in time of the quantity $(I_1^2 + s^2I_0^2)(t) = \|\varphi_x\|^2 + s^2\|\varphi\|^2$, i.e., the evolution of kinematic perturbation in measure of the space H_2. The initial value $(I_1^2 + s^2I_0^2)(0)$ is known and can be regarded as a missing initial condition in the integral sense.

Let us follow the line of reasoning of the paper [8], carrying out the generalization of results to the case of unsteady basic flow, which is of the main interest in the present paper.

2.1. Kinematic Boundary Conditions

Consider the kinematic conditions (1.4) which are the simplest ones in their mathematical formulation, or the adhesion conditions. All summands with the substitutions at the points $x = 0$ and $x = 1$ in (2.2) vanish, and thus the following equation holds for the real parts:

$$\frac{1}{2}\frac{d}{dt}(I_1^2 + s^2I_0^2) = s\int_0^1 v^\circ_x(\varphi_x\bar\varphi)_{**}\,dx - \frac{1}{\mathrm{Re}}(I_2^2 + 2s^2I_1^2 + s^4I_0^2), \tag{2.5}$$

which implies the following chain of inequalities (one of which is the Schwarz inequality in H_2):

$$\frac{1}{2}\frac{d}{dt}(I_1^2 + s^2I_0^2) \leqslant qsI_1I_0 - \frac{\Lambda^2}{\mathrm{Re}}(I_1^2 + s^2I_0^2) \leqslant \Big(\frac{q}{2} - \frac{\Lambda^2}{\mathrm{Re}}\Big)(I_1^2 + s^2I_0^2) \leqslant a(t)(I_1^2 + s^2I_0^2), \tag{2.6}$$

where

$$q(t) = \sup_{0<x<1} |v_x^\circ(x,t)|, \quad \Lambda^2[\varphi; s](t) = \frac{I_2^2 + s^2 I_1^2}{I_1^2 + s^2 I_0^2} + s^2, \tag{2.7}$$

$$a(t) = \frac{q}{2} - \frac{1}{\mathrm{Re}} \inf_{\varphi; s} \Lambda^2. \tag{2.8}$$

The minimization of the quadratic functional $\Lambda^2[\varphi; s]$ is carried out both over all functions satisfying the conditions in (1.4) and over the semiaxis of wave numbers $s > 0$. In [8], using the analysis of Fourier series, the author proved that this minimum is achieved at the function

$$\varphi_{\min} = \varphi_0 \left(\sin \pi x - \frac{1}{3} \sin 3\pi x \right), \quad \varphi_0 \in C, \tag{2.9}$$

$$I_0^2[\varphi_{\min}] = \frac{5}{9} |\varphi_0|^2, \quad I_1^2[\varphi_{\min}] = \pi^2 |\varphi_0|^2, \quad I_2^2[\varphi_{\min}] = 5\pi^4 |\varphi_0|^2,$$

$$\Lambda^2[\varphi_{\min}; s] = \frac{(5\pi^2 + s^2)\pi^2}{\pi^2 + 5s^2/9} + s^2, \tag{2.10}$$

and the wave number $s_{\min}^2 = 3\pi^2/5$:

$$\inf_{\varphi; s} \Lambda^2 = \inf_{s>0} \Big(\Lambda^2[\varphi_{\min}; s] \Big) = \frac{24\pi^2}{5}. \tag{2.11}$$

Thus, for the kinematic boundary conditions (1.4), the function $a(t)$ (2.8) of time is

$$a(t) = \frac{q(t)}{2} - \frac{24\pi^2}{5\mathrm{Re}}.$$

The left and right ends of the chain (2.6) lead immediately to the exponential upper bound

$$(I_1^2 + s^2 I_0^2)(t) \leqslant (I_1^2 + s^2 I_0^2)(0) \exp \int_0^t 2a(\tau)\, d\tau = (I_1^2 + s^2 I_0^2)(0) \exp \Big[\int_0^t q(\tau)\, d\tau - \frac{48\pi^2}{5\mathrm{Re}}\, t \Big]. \tag{2.12}$$

In the case of steady basic motion, we have $q(t) = q_0 = \mathrm{const}$, and the exponential function in (2.12) tends to zero as $t \to \infty$ if $q_0 \,\mathrm{Re} < 48\pi^2/5$. This corresponds to decaying the initial perturbations in measure of H_2, or to "stability in the integral sense." Hence, the true critical Reynolds number Re_* is certainly greater than $48\pi^2/(5q_0)$.

2.2. Force Boundary Condition on One of the Two Straight Boundaries

Let us keep the adhesion condition on the line $x = 0$ and define a shear stress $\sigma^\circ(t)$ on the line $x = 1$, which corresponds the requirements in (1.5). Here the characteristic velocity V entering the Reynolds number is chosen to be equal to $\sqrt{\sigma^\circ/\rho}$ (σ° stands for the characteristic stress). The manipulations of Subsection 2.1 remain valid, up to bounds (2.6)–(2.8); however, the minimization of the quadratic functional Λ^2 is to be carried out on the class of functions satisfying new boundary conditions. In this case, the minimizing function for every s is [8]

$$\varphi_{\min} = \varphi_0 \left(\sin \pi x - \frac{1}{2} \sin 2\pi x \right), \quad \varphi_0 \in C, \tag{2.13}$$

$$I_0^2[\varphi_{\min}] = \frac{5}{8} |\varphi_0|^2, \quad I_1^2[\varphi_{\min}] = \pi^2 |\varphi_0|^2, \quad I_2^2[\varphi_{\min}] = \frac{5}{2} \pi^4 |\varphi_0|^2,$$

$$\Lambda^2[\varphi_{\min}; s] = \frac{(5\pi^2 + 2s^2)\pi^2}{2\pi^2 + 5s^2/4} + s^2. \tag{2.14}$$

The lower bound of expression (2.14) is attained in the limit as $s \to 0$, and this bound is equal to $5\pi^2/2$.

We have $a(t) = q(t)/2 - 5\pi^2/(2\mathrm{Re})$ in (2.8), and thus the following bound holds:

$$(I_1^2 + s^2 I_0^2)(t) \leqslant (I_1^2 + s^2 I_0^2)(0) \exp\Big[\int_0^t q(\tau)\, d\tau - \frac{5\pi^2}{\mathrm{Re}}\, t\Big]. \tag{2.15}$$

In the steady case, this bound implies a lower bound for the true critical Reynolds number, $\mathrm{Re}_* > 5\pi^2/q_0$.

3. UNSTEADY BASIC MODE

Let us dwell on several modifications in time, which are typical in practical applications, of the velocity profile $v^\circ(x,t)$ characterizing the basic shear flow. Let us not dwell on the problem concerning the initial boundary value problems in the layer Ω whose solutions are the profiles in question, all the more, because this issue can always be bypassed by involving fictitious mass forces.

3.1. Power-Law Acceleration and Deceleration

Let

$$v^\circ = V^\circ(x)(t+t_0)^\gamma, \quad q(t) = q_0(t+t_0)^\gamma, \quad q_0 = \sup_{0<x<1} |V^{\circ\prime}(x)|, \quad t_0 > 0. \tag{3.1}$$

Then the exponents in (2.12) and (2.15) are of the form

$$\int_0^t a(\tau)\, d\tau = \frac{q_0}{2(\gamma+1)}\big[(t+t_0)^{\gamma+1} - t_0^{\gamma+1}\big] - \frac{\lambda\pi^2}{\mathrm{Re}}\, t, \qquad \gamma \neq -1,$$

$$\int_0^t a(\tau)\, d\tau = \frac{q_0}{2} \ln\frac{t+t_0}{t_0} - \frac{\lambda\pi^2}{\mathrm{Re}}\, t, \qquad \gamma = -1,$$

where λ is equal to $24/5$ and $5/2$ for the cases in which the boundary conditions are given as in Subsection 2.1 and 2.2, respectively.

For $\gamma < 0$ (deceleration of the layer), the exponents tend as $t \to \infty$ to $-\infty$ for every Re, which means the exponential decay of the perturbations; $\gamma = 0$ is the steady case in which the stability depends on the value of Re (see Section 2); for $\gamma > 0$ (acceleration of the layer), inequalities (2.12) and (2.15) become bounds for the growth of perturbations.

3.2. Oscillatory Mode

Let the basic motion describe harmonic oscillations of the layer Ω with the frequency ω:

$$v^\circ = V^\circ(x)\sin\omega t, \qquad q(t) = q_0|\sin\omega t|. \tag{3.2}$$

Since

$$\int_0^{t_n} |\sin\omega t|\, d\tau = \frac{2n}{\omega}, \quad t_n = \frac{\pi n}{\omega}, \quad n = 1, 2, \dots,$$

we arrive at the following values of the exponent at the point t_n,

$$\int_0^{t_n} a(\tau)\, d\tau = \frac{2nq_0}{\omega} - \frac{\lambda\pi^2}{\mathrm{Re}}\, t_n = \Big(\frac{2q_0}{\pi} - \frac{\lambda\pi^2}{\mathrm{Re}}\Big)t_n.$$

For $q_0\mathrm{Re} < \lambda\pi^3/2$, we have an exponential decay of the initial perturbations. Note that the final bound of the product $q_0\mathrm{Re}$ does not contain the frequency ω.

3.3. A Mode Similar to a Diffusion of a Vortex Layer

Consider the basic motion with the profile

$$v^\circ = \int_0^t \operatorname{erfc}\left(\frac{x}{2}\sqrt{\frac{\mathrm{Re}}{t-\tau}}\right) dU(\tau), \quad \operatorname{erfc} y = 1 - \frac{2}{\sqrt{\pi}} \int_0^y \mathrm{e}^{-\zeta^2}\, d\zeta, \tag{3.3}$$

where $\operatorname{erfc} y$ stands for the complementary error function. The solution (3.3) is exact in the diffusion problem of a vortex layer in the viscous half-plane $x > 0$ when the boundary $x = 0$ moves with the given longitudinal velocity $U(t)$. This velocity is such that $U \equiv 0$ for $t \in (-\infty; -0)$, and it can have discontinuities of the first kind for $t \in (+0; \infty)$ that admit the existence of the Stieltjes integral in (3.3). Here $dU(t)$ is to be understood as

$$dU(t) = \Bigl(\dot U(t) + \sum_i [U]_i \delta(t - t_i)\Bigr)\, dt,$$

where $[U]_i$ is the jump of the function U at the discontinuity point t_i.

Naturally, the presence of the boundary $x = 1$ makes the solution (3.3) in the layer Ω somewhat artificial, because it makes it necessary that this boundary moves (for an already defined function $U(\tau)$) in a strictly definite way,

$$v^\circ(1, t) = \int_0^t \operatorname{erfc}\sqrt{\frac{\mathrm{Re}}{4(t-\tau)}}\, dU(\tau).$$

Let us evaluate $q(t)$ according to (2.7) and (3.3),

$$q(t) = \sup_{0<x<1} \left| -\sqrt{\frac{\mathrm{Re}}{\pi}} \int_0^t \exp\left(-\frac{x^2 \mathrm{Re}}{4(t-\tau)}\right) \frac{dU(\tau)}{\sqrt{t-\tau}} \right| = \sqrt{\frac{\mathrm{Re}}{\pi}} \int_0^t \frac{|dU(\tau)|}{\sqrt{t-\tau}}. \tag{3.4}$$

Then

$$\int_0^t a(\tau)\, d\tau = \sqrt{\frac{\mathrm{Re}}{4\pi}} \int_0^t \int_0^\tau \frac{|dU(\tilde\tau)|}{\sqrt{\tau - \tilde\tau}}\, d\tau - \frac{\lambda \pi^2}{\mathrm{Re}}\, t. \tag{3.5}$$

Prescribing the function U determines the behavior of the expression (3.5) as $t \to \infty$. For example, in the self-similar case of classical diffusion of the vortex layer in which $U(t) = U_0 h(t)$, where $h(t)$ stands for the Heaviside function, $dh(t) = \delta(t)\, dt$, and $U_0 > 0$, we can see by (3.4) and (3.5) that

$$q(t) = \sqrt{\frac{\mathrm{Re}}{\pi t}}\, U_0; \quad \int_0^t a(\tau)\, d\tau = \sqrt{\frac{\mathrm{Re}\, t}{2\pi}} - \frac{\lambda\pi^2}{\mathrm{Re}}\, t \quad \longrightarrow \ -\infty, \quad t \to \infty \quad \forall\ \mathrm{Re}.$$

4. REDUCIBILITY OF THE THREE-DIMENSIONAL PICTURE OF PERTURBATIONS TO THE TWO-DIMENSIONAL ONE

The Orr–Sommerfeld equation (1.1) itself and its unsteady generalization (1.9) are derived under the assumption that the picture of perturbations imposed on the basic flow is two-dimensional and belongs to the plane (x_1, x_2) of this flow. What are cases in which actual three-dimensional perturbations can be reduced to two-dimensional ones? Is it possible that this kind of consideration of three-dimensionality can reduce the stability resources? An answer to these questions and some other ones is given by the Squire theorem [2], which is classical in the theory of hydrodynamic stability, and diverse generalizations of this theorem [14].

In the three-dimensional layer $\Omega' = \Omega \times (-\infty; \infty)$, we consider the linearized Navier–Stokes system with respect to the perturbations of the pressure p and the components of the velocity vector v_k,

$$-p_{,k} + \frac{1}{\mathrm{Re}}\, v_{k,jj} = \frac{\partial v_k}{\partial t} + v_j^\circ v_{k,j} + v_{k,j}^\circ v_j, \quad k = 1, 2, 3, \quad v_j^\circ = v^\circ(x_2, t)\delta_{j1}, \tag{4.1}$$

$$v_{j,j} = 0. \tag{4.2}$$

Let us represent four functions $p(x_1, x_2, x_3, t)$ and $v_k(x_1, x_2, x_3, t)$ in the form

$$p = \tilde{p}(x_2, t)\mathrm{e}^{i(s_1x_1+s_3x_3)}, \qquad v_k = \tilde{v}_k(x_2, t)\mathrm{e}^{i(s_1x_1+s_3x_3)}, \tag{4.3}$$

where $s_1 > 0$ and $s_3 \geqslant 0$ are the wave numbers of the harmonics along the axes x_1 and x_3. After substituting (4.3) into (4.1) and (4.2), multiplying the first equation (4.1) by s_1 and the third by s_3, adding the results and dividing by s_1, we write out the system of three equations with respect to $\tilde{q}(x_2, t)$, $\tilde{u}_1(x_2, t)$, and $\tilde{u}_2(x_2, t)$,

$$\begin{gathered} -is\tilde{q} + \frac{s}{s_1\mathrm{Re}}\left(-s^2\tilde{u}_1 + \tilde{u}_{1,xx}\right) = \frac{s}{s_1}\,\tilde{u}_{1,t} + isv^\circ\tilde{u}_1 + v^\circ_x\tilde{u}_2, \\ -\tilde{q}_x + \frac{s}{s_1\mathrm{Re}}\left(-s^2\tilde{u}_2 + \tilde{u}_{2,xx}\right) = \frac{s}{s_1}\,\tilde{u}_{2,t} + isv^\circ\tilde{u}_2, \\ is\tilde{u}_1 + \tilde{u}_{2,x} = 0, \end{gathered} \tag{4.4}$$

where we use the notation

$$s = \sqrt{s_1^2 + s_3^2}, \quad \tilde{u}_1 = \frac{1}{s}(s_1\tilde{v}_1 + s_3\tilde{v}_3), \quad \tilde{u}_2 = \tilde{v}_2, \quad \tilde{q} = \frac{s}{s_1}\,\tilde{p}, \quad x \equiv x_2. \tag{4.5}$$

Instead of (4.3), let us now study the two-dimensional picture of perturbations in the plane (x_1, x_2) (this picture is similar to (1.8)),

$$p = q(x_2, t)\mathrm{e}^{isx_1}, \quad v_1 = u_1(x_2, t)\mathrm{e}^{isx_1}, \quad v_2 = u_2(x_2, t)\mathrm{e}^{isx_1}, \tag{4.6}$$

and substitute this into the original system (4.1), (4.2) by setting $v_3 \equiv 0$,

$$\begin{gathered} -isq + \frac{1}{\mathrm{Re}}\left(-s^2u_1 + u_{1,xx}\right) = u_{1,t} + isv^\circ u_1 + v^\circ_x u_2, \\ -q_x + \frac{1}{\mathrm{Re}}\left(-s^2u_2 + u_{2,xx}\right) = u_{2,t} + isv^\circ u_2, \\ isu_1 + u_{2,x} = 0. \end{gathered} \tag{4.7}$$

Systems of equations (4.4) and (4.7) (each with its own triple of unknowns) are very similar mathematically. The idea of Squire transform is in the precise comparison of the systems.

4.1. Steady Basic Motion

Since v° does not depend on t explicitly in this case, it follows that the only difference between systems (4.4) and (4.7) is in the presence of the coefficient s/s_1 in (4.4) in the summands with the Reynolds number and with the partial derivative with respect to time. Using the formal scaling, $\widetilde{\mathrm{Re}} = s_1\mathrm{Re}/s$ and $\tilde{t} = s_1t/s$, we see that there is a connection between the solutions of systems (4.4) and(4.7),

$$\tilde{q}(x_2, \tilde{t}; \widetilde{\mathrm{Re}}) = q(x_2, t; \mathrm{Re}), \quad \tilde{u}_k(x_2, \tilde{t}; \widetilde{\mathrm{Re}}) = u_k(x_2, t; \mathrm{Re}), \quad k = 1, 2. \tag{4.8}$$

Note that $\mathrm{Re} \geqslant \widetilde{\mathrm{Re}}$ and $t \geqslant \tilde{t}$ by (4.5). The inequality $\mathrm{Re} \geqslant \widetilde{\mathrm{Re}}$ means that the harmonic of a perturbation propagating at an angle of the plane $(x_1; x_2)$ behaves just as its projection to the plane with lesser Reynolds number. Thus, the critical Reynolds number Re_*, if it exists, is minimal for the two-dimensional perturbations in the plane $(x_1; x_2)$. Hence, these very perturbations are most dangerous in the sense of stability. This is the statement of the Squire theorem.

The inequality $t \geqslant \tilde{t}$ shows the extension in time for the behavior of solutions of system (4.4) as compared with (4.7). Since the stability of the basic flow is considered on the infinite time interval and depends on the behavior of the picture of perturbation as $t \to \infty$, this scaling does not influence in the fact of stability itself.

One should keep in mind that the problem of reducibility of three-dimensional perturbations to two-dimensional ones is to be investigated as applied not only to systems (4.4) and (4.7), but also to boundary value problems posed for these systems. For example, the classical Squire theorem is formulated for the Orr–Sommerfeld problem itself, i.e., for the case in which the adhesion conditions are chosen. As is proved in [8], if, to be definite, on one of the straight parts of the boundary, we accept the adhesion conditions and, on the other part, we either pose the same conditions $v_1 = v_2 = v_3 = 0$ or the static conditions $\sigma_{12} = \sigma_{32} = 0$ and $v_2 = 0$, then one can implement the Squire transform. However, this transform cannot be implemented in general in the case of mixed conditions on the second straight boundary, $\sigma_{12} = 0$ and $v_2 = v_3 = 0$, which admit slippage, and the corresponding three-dimensional picture of perturbation cannot be reduced to a two-dimensional one.

4.2. Unsteady Basic Flow

If the velocity profile v° in systems (4.4) and (4.7) depends on t explicitly, then the relationships concerning solutions of (4.8) fail to hold, as well as the corresponding conclusions, and the Squire theorem is not valid here. This means that the dominating unstable harmonics of the perturbations imposed on unsteady plane-parallel shear flows can be placed at an angle to the plane of the basic shear.

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