Comparison of different methodologies leading to a generalized elasticity theory for modeling of the size effect

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Abstract

Miniaturization in the production technology necessitates more accurate modeling of structures in the micrometer length scales. The well-known theory of elasticity shows inadequacies when trying to mimic experimental observations. Hence, it needs to be generalized. There are two different ways in continuum mechanics for the formulation of elasticity: Rational mechanics and classical analytical mechanics. Both methods can be used to study and generalize *traditional* elasticity theory. In this work we aim at a comparison of these methods in order to obtain more insight into their pros and cons as well as their limitations.

1 Introduction

Traditional elasticity theory of continuum mechanics fails to model structures on the micrometer length scale with sufficient accuracy. However, nowadays, for example in micro-electro-mechanical systems (MEMS), modeling and simulation of small scale designs are of paramount importance. Many experimental observations over the last decades—see for example [6], [7], [5]—indicate a size effect showing up at the micrometer length scale. By scaling down the dimensions of a structure, the deformation starts to deviate from the expected results calculated by traditional elasticity theory. This behavior is referred to as the size effect. As justified by many experimental observations we need a generalization of elasticity theory. There are, indeed, many propositions about possible generalizations in the literature. The different theories can be grouped into two methods.

Rational mechanics can be used to obtain a generalization of the elasticity theory. In addition to the balance of linear momentum, the balances of angular momentum and spin are used in order to capture the effects introduced by materials showing a size effect. In this work, we give a brief outline of the method based on rational mechanics, which has already been discussed in [3] and [2].

Classical analytical mechanics delivers a straightforward way for generalizing elasticity theory. Starting from the principle of least action, we can incorporate the effects

necessary for modeling the size effect. We will also outline the method stemming from classical analytical mechanics as done in [1].
The two different methodologies lead to weak forms that allow us to compute the size effect as observed in experiments. Both methods deliver the same deformation behavior as observed in experiments. However, the methods are inherently different. Hence, in this work, we want to compare the methods and find out their analogies as well as dissimilarities.

2 Rational mechanics

Throughout the paper we use the standard notation of continuum mechanics. Summation is implied if double indices appear in a product. In rational mechanics we start axiomatically with the balance equations. They are given at the current time for a material system, where the totality of particles compiling the continuum body remains the same. The continuum body, $\mathcal{B}$, consists of the same particles and the balances of mass, linear and angular momenta are:

$$\begin{aligned} \left(\int_{\mathcal{B}} \rho \,\mathrm{d}v\right)^{\cdot} &= 0 \ , \\ \left(\int_{\mathcal{B}} p_i^{\text{lin.}} \,\mathrm{d}v\right)^{\cdot} &= \int_{\partial\mathcal{B}} \sigma_{ji} \,\mathrm{d}a_j + \int_{\mathcal{B}} \rho f_i \,\mathrm{d}v \ , \\ \left(\int_{\mathcal{B}} p_i^{\text{ang.}} \,\mathrm{d}v\right)^{\cdot} &= \int_{\partial\mathcal{B}} \alpha_{ji} \,\mathrm{d}a_j + \int_{\mathcal{B}} \rho z_i \,\mathrm{d}v \ , \end{aligned} \tag{1}$$

respectively. The mass density, ρ, is the mass per unit volume. The linear and angular momenta densities read:

$$p_i^{\text{lin.}} = \rho v_i \ , \quad p_i^{\text{ang.}} = \rho a_i \ , \tag{2}$$

where the specific linear momentum (per mass) or velocity, v_i, and the specific angular momentum, a_i, are the sought variables. On the right-hand sides of the balance equations the fluxes and supplies of momenta appear. The flux of linear momentum, σ_{ji}, is called CAUCHY's stress. The flux term of angular momentum, α_{ji}, has no particular name. The supply of the linear momentum, f_i, is the specific body force, for example the gravitational force per mass (acceleration). The supply of the angular momentum, z_i, is also a volumetric effect, leading to a change in the angular momentum.
In solid mechanics we transform the balance equations onto the reference frame from where the particles can be tracked. For example, we can use the initial positions of particles, X_i, as the reference frame such that X_i denotes the particles. This configuration is called a LAGRANGEan frame and by employing tensor calculus we obtain the balance of mass and momenta in the LAGRANGEan frame, locally,

$$\begin{aligned} \rho_0 &= \rho J \ , \\ \rho J \frac{\partial v_i}{\partial t} &= \frac{\partial P_{ji}}{\partial X_j} + \rho J f_i \ , \\ \rho J \frac{\partial a_i}{\partial t} &= \frac{\partial A_{ji}}{\partial X_j} + \rho J z_i \ , \end{aligned} \tag{3}$$

with the fluxes of linear and angular momenta in the LAGRANGEan frame:

$$P_{ji} = (\boldsymbol{F}^{-1})_{jk} J\sigma_{ki} \ , \quad A_{ji} = (\boldsymbol{F}^{-1})_{jk} J\alpha_{ki} \ . \tag{4}$$

The deformation gradient, F_{ij}, is the mapping between the current and reference (here initial) frame and J is its determinant:

$$F_{ij} = \frac{\partial x_i}{\partial X_j} = \frac{\partial u_i}{\partial X_j} + \delta_{ij} \ , \quad J = \det(\boldsymbol{F}) \ . \tag{5}$$

The balance of mass is an equation instead of a differential equation, so we just use it instead of solving it. From the solution of the following differential equations:

$$\begin{aligned} \rho_0 \frac{\partial v_i}{\partial t} &= \frac{\partial P_{ji}}{\partial X_j} + \rho_0 f_i \ , \\ \rho_0 \frac{\partial a_i}{\partial t} &= \frac{\partial A_{ji}}{\partial X_j} + \rho_0 z_i \ , \end{aligned} \tag{6}$$

we obtain v_i and a_i as functions of (X_i, t). We may introduce another quantity, called spin, s_i, as follows:

$$a_i = s_i + \epsilon_{ijk} X_j v_k \ . \tag{7}$$

In rigid body mechanics the second part of this relation is well-known. However, in continuum mechanics spin is an intrinsic property of a *polar* material, which we fail to observe directly, like internal energy. The best example is a nematic fluid (as in a liquid crystal display) where the change of spin varies the interaction of the matter with light. For non-polar materials spin vanishes, $s_i = 0$. The LEVI-CIVITA symbol, ϵ_{ijk}, allows to determine the cross product between the particle's position X_i and its velocity v_i. It is called the moment of linear momentum in the LAGRANGEan frame. Since we have to fulfill all balance equations simultaneously, we can multiply the balance of linear momentum with $\epsilon_{kli} X_l$ and then subtract from the balance of angular momentum to obtain the balance of spin exclusively:

$$\begin{aligned} \rho_0 \frac{\partial s_k}{\partial t} = \frac{\partial}{\partial X_j}\Big(A_{jk} - \epsilon_{kli} X_l P_{ji}\Big) + P_{ji} \frac{\partial \epsilon_{kli} X_l}{\partial X_j} + \rho_0 (z_k - \epsilon_{kli} X_l f_i) \ , \\ \rho_0 \frac{\partial s_k}{\partial t} = \frac{\partial \mu_{jk}}{\partial X_j} + \epsilon_{kji} P_{ji} + \rho_0 l_k \ , \end{aligned} \tag{8}$$

with the flux of spin and the specific supply of spin,

$$\mu_{jk} = A_{jk} - \epsilon_{kli} X_l P_{ji} \ , \quad l_k = z_k - \epsilon_{kli} X_l f_i \ , \tag{9}$$

respectively. In case of a non-polar material the spin vanishes, $s_i = 0$. Then we multiply the spin balance for a non-polar material by $\frac{1}{2}\epsilon_{mnk}$ and employ the identity $\epsilon_{mnk}\epsilon_{kji} = \delta_{mj}\delta_{ni} - \delta_{mi}\delta_{nj}$, such that we obtain

$$\begin{aligned} 0 &= \frac{1}{2}\epsilon_{mnk} \frac{\partial \mu_{jk}}{\partial X_j} + \frac{1}{2}\epsilon_{mnk}\epsilon_{kji} P_{ji} + \frac{1}{2}\epsilon_{mnk}\rho_0 l_k \ , \\ 0 &= \frac{\partial \mu_{jmn}}{\partial X_j} + \frac{1}{2}\Big(P_{mn} - P_{nm}\Big) + \rho_0 l_{mn} \ , \\ -\frac{\partial \mu_{jmn}}{\partial X_j} &- \frac{1}{2}\Big(P_{mn} - P_{nm}\Big) - \rho_0 l_{mn} = 0 \end{aligned} \tag{10}$$

with:

$$\mu_{jmn} = \frac{1}{2}\mu_{jk}\epsilon_{kmn} \ , \quad l_{mn} = \frac{1}{2}l_k\epsilon_{kmn} \ . \tag{11}$$

By using a shorthand notation with antisymmetric brackets we acquire:

$$-\frac{\partial \mu_{jmn}}{\partial X_j} - P_{[mn]} - \rho_0 l_{mn} = 0 \ . \tag{12}$$

The tensor of rank three, μ_{jmn}, is often called the couple stress. Since the spin vanishes for a non-polar material, the only unknown to be calculated that remains is the displacement. Its rate is velocity such that the balance of linear momentum reads:

$$\rho_0 \frac{\partial^2 u_i}{\partial t^2} - \frac{\partial P_{ji}}{\partial X_j} - \rho_0 f_i = 0 \ . \tag{13}$$

The nominal stress tensor, P_{ji}, as well as couple stress, μ_{jmn}, need to be defined by means of the displacement or its gradients in order to close Eqs. (12), (13). We refer to [4] for the detailed derivation of the following constitutive equations for an isotropic material:

$$S_{ij} = C_{ijkl}E_{kl} \ , \quad \mu_{ijk} = D_{ijklmn}E_{lm,n} \ , \tag{14}$$

where the second PIOLA-KIRCHHOFF, S_{ij}, and GREEN-LAGRANGE strains, E_{ij}, are given by

$$S_{ij} = (\boldsymbol{F}^{-1})_{jk}P_{ik} \ , \quad E_{ij} = \frac{1}{2}\big(F_{ki}F_{kj} - \delta_{ij}\big) \ . \tag{15}$$

We start using a simplified notation for the derivative in space:

$$(\cdot)_{,i} = \frac{\partial(\cdot)}{\partial X_i} \ . \tag{16}$$

For isotropic materials the stress depends only on strain and the couple stress depends only on strain gradients. The dependency is such that the stiffness tensor, C_{ijkl}, and the material tensor, D_{ijklmn}, is expanded as follows:

$$\begin{gathered} C_{ijkl} = c_1\delta_{ij}\delta_{kl} + c_2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \ , \\ D_{ijklmn} = c_3(\delta_{ij}\delta_{kl}\delta_{mn} + \delta_{in}\delta_{jk}\delta_{lm} + \delta_{ij}\delta_{km}\delta_{ln} + \delta_{ik}\delta_{jn}\delta_{lm}) + c_4\delta_{ij}\delta_{kn}\delta_{ml} + \\ + c_5(\delta_{ik}\delta_{jl}\delta_{mn} + \delta_{im}\delta_{jk}\delta_{ln} + \delta_{ik}\delta_{jm}\delta_{ln} + \delta_{il}\delta_{jk}\delta_{mn}) + c_6(\delta_{il}\delta_{jm}\delta_{kn} + \delta_{im}\delta_{jl}\delta_{kn}) + \\ + c_7(\delta_{il}\delta_{jn}\delta_{mk} + \delta_{im}\delta_{jn}\delta_{lk} + \delta_{in}\delta_{jl}\delta_{km} + \delta_{in}\delta_{jm}\delta_{kl}) \ . \end{gathered} \tag{17}$$

For a linear, isotropic, homogeneous material, the seven material parameters, $c_\times$, are constant values. It is also beneficial to note the following relation for the so-called stored energy density:

$$w = \int S_{ij}\,\mathrm{d}E_{ji} + \int \mu_{ijk}\,\mathrm{d}E_{kj,i} = \frac{1}{2}C_{ijkl}E_{kl}E_{ji} + \frac{1}{2}D_{ijklmn}E_{lm,n}E_{kj,i} \ , \tag{18}$$

which holds, because C_{ijkl} and D_{ijklmn} are constants for a linear material. Obviously, the following relations result:

$$S_{ij} = \frac{\partial w}{\partial E_{ji}} \ , \quad \mu_{ijk} = \frac{\partial w}{\partial E_{kj,i}} \ . \tag{19}$$

For a numerical computation we need a so-called weak form. In order to obtain the weak form for the whole continuum body, $\mathcal{B}_0$, we multiply the balance Eqs. (12), (13) written as a residuals by the so-called test functions, δu_i, $\delta u_{n,m}$, respectively,

$$\int_{\mathcal{B}_0} \left(\left(\rho_0 \frac{\partial^2 u_i}{\partial t^2} - P_{ji,j} - \rho_0 f_i \right) \delta u_i - \left(\mu_{jmn,j} + P_{[mn]} + \rho_0 l_{mn} \right) \delta u_{n,m} \right) \mathrm{d}V = 0 \ . \tag{20}$$

We discretize in time by using a Euler backwards finite difference scheme,

$$\begin{aligned} \int_{\mathcal{B}_0} \Bigg(\rho_0 \frac{u_i - 2u_i^0 + u_i^{00}}{\Delta t^2} \delta u_i - P_{ji,j} \delta u_i - \rho_0 f_i \delta u_i - \mu_{jmn,j} \delta u_{n,m} - \\ - P_{[mn]} \delta u_{n,m} - \rho_0 l_{mn} \delta u_{n,m} \Bigg) \mathrm{d}V = 0 \ , \end{aligned} \tag{21}$$

where u_i^0 denotes the value from the last time step and u_i^{00} indicates the value from the two time steps before. According to the Galerkin procedure in the finite element method, we choose the test functions, δu_i, from the same space as u_i. Stress and couple stress have already a derivative in u_i and $u_{i,j}$ such that another differentiation with respect to X_i increases the necessity of differentiability. We can weaken this fact by "shifting" the derivative to the test functions by employing integrations by parts as follows:

$$\begin{aligned} \int_{\mathcal{B}_0} \Bigg(\rho_0 \frac{u_i - 2u_i^0 + u_i^{00}}{\Delta t^2} \delta u_i + P_{ji} \delta u_{i,j} - \rho_0 f_i \delta u_i + \mu_{jmn} \delta u_{n,mj} - \\ - P_{[mn]} \delta u_{n,m} - \rho_0 l_{mn} \delta u_{n,m} \Bigg) \mathrm{d}V - \int_{\partial \mathcal{B}_0} \left(P_{ji} \delta u_i + \mu_{jmn} \delta u_{n,m} \right) N_j \, \mathrm{d}A = 0 \ . \end{aligned} \tag{22}$$

The boundary terms shall be known. The force per area, $\hat{t}_i = N_j P_{ji}$, is called the traction vector. Analogously, we can call the moment per area, $\hat{m}_{mn} = N_j \mu_{jmn}$, the torsion vector. Finally, the weak form for a non-polar, isotropic, linear elastic material reads:

$$\begin{aligned} \mathrm{Form} = \int_{\mathcal{B}_0} \Bigg(\rho_0 \frac{u_i - 2u_i^0 + u_i^{00}}{\Delta t^2} \delta u_i + P_{ji} \delta u_{i,j} - \rho_0 f_i \delta u_i + \mu_{jmn} \delta u_{n,mj} + \\ - P_{[mn]} \delta u_{n,m} - \rho_0 l_{mn} \delta u_{n,m} \Bigg) \mathrm{d}V - \int_{\partial \mathcal{B}_0} \left(\hat{t}_i \delta u_i + \hat{m}_{mn} \delta u_{n,m} \right) \mathrm{d}A \ . \end{aligned} \tag{23}$$

By using the finite element method we can compute the displacement, $u_i(X_j, t)$, satisfying the balance of momenta in the Lagrangean frame.

3 Classical analytical mechanics

In the case of classical analytical mechanics we start from a LAGRANGEan density, $\mathcal{L}$, and define an action:

$$\mathcal{S} = \int_\Omega \mathcal{L}\,\mathrm{d}\Sigma\ , \quad \mathrm{d}\Sigma = \mathrm{d}x_1\,\mathrm{d}x_2 \ldots \mathrm{d}x_m\ , \tag{24}$$

in m-dimensional space. In mechanics the space, x_ν, is space-time, $x_\nu = \{X_1, X_2, X_3, t\}$, so that the infinitesimal volume element reads $\mathrm{d}\Sigma = \mathrm{d}V\,\mathrm{d}t$. The axiom is the principle of least action stating the postulate that the action remains the same, if we transform the primitive variables. Primitive variables are displacements. In the application of the traditional elasticity theory, the LAGRANGEan density depends on the primitive variables and its derivatives:

$$\mathcal{L} = \mathcal{L}\Big(x_\nu, u_i, u_{i,\nu}\Big)\ . \tag{25}$$

The comma notation means a partial derivative with respect to space-time:

$$u_{i,\nu} = \frac{\partial u_i}{\partial x_\nu} = \begin{pmatrix} \dfrac{\partial u_i}{\partial x_j} \\ \dfrac{\partial u_i}{\partial t} \end{pmatrix}\ . \tag{26}$$

For the next higher gradient theory, we incorporate the second derivatives, such that the LAGRANGEan density becomes:

$$\mathcal{L} = \mathcal{L}\big(x_\nu, u_i, u_{i,\nu}, u_{i,\nu\mu}\big)\ . \tag{27}$$

The principle of least action states:

$$\delta\mathcal{S} = \int_\Omega \mathcal{L}'\,\mathrm{d}\Sigma - \int_\Omega \mathcal{L}\,\mathrm{d}\Sigma = 0\ , \tag{28}$$

where the transformed LAGRANGEan density is:

$$\mathcal{L}' = \mathcal{L}\Big(x_\nu, u_i', u_{i,\nu}', u_{i,\nu\mu}'\Big)\ , \tag{29}$$

with arbitrary variations made linear by using a small constant number, ε, as follows:

$$u_i' = u_i + \varepsilon\delta u_i\ , \quad u_{i,\nu}' = u_{i,\nu} + \varepsilon\delta u_{i,\nu}\ , \quad u_{i,\nu\mu}' = u_{i,\nu\mu} + \varepsilon\delta u_{i,\nu\mu}\ . \tag{30}$$

Now by expanding $\mathcal{L}'$ around $\mathcal{L}$ we obtain:

$$\mathcal{L}' = \mathcal{L} + \frac{\partial\mathcal{L}}{\partial u_i}\varepsilon\delta u_i + \frac{\partial\mathcal{L}}{\partial u_{i,\nu}}\varepsilon\delta u_{i,\nu} + \frac{\partial\mathcal{L}}{\partial u_{i,\nu\mu}}\varepsilon\delta u_{i,\nu\mu}\ . \tag{31}$$

Hence, the principle of least action results in:

$$\delta\mathcal{S} = \int_\Omega \left(\frac{\partial\mathcal{L}}{\partial u_i}\delta u_i + \frac{\partial\mathcal{L}}{\partial u_{i,\nu}}\delta u_{i,\nu} + \frac{\partial\mathcal{L}}{\partial u_{i,\nu\mu}}\delta u_{i,\nu\mu}\right)\mathrm{d}\Sigma = 0\ , \tag{32}$$

since ε is constant in space, x_ν. Moreover, the space, x_ν, is space-time in the LAGRANGEan frame, $\{X_i, t\}$, thus, the principle of least action asserts:

$$\int_\tau \int_{\mathcal{B}_0} \left(\frac{\partial \mathcal{L}}{\partial u_i} \delta u_i + \frac{\partial \mathcal{L}}{\partial \dot{u}_i} \delta \dot{u}_i + \frac{\partial \mathcal{L}}{\partial u_{i,j}} \delta u_{i,j} + \frac{\partial \mathcal{L}}{\partial \ddot{u}_i} \delta \ddot{u}_i + \right. \\ \left. + \frac{\partial \mathcal{L}}{\partial \dot{u}_{i,j}} \delta \dot{u}_{i,j} + \frac{\partial \mathcal{L}}{\partial u_{i,jk}} \delta u_{i,jk} \right) \mathrm{d}V \, \mathrm{d}t = 0 \ . \tag{33}$$

By using integration by parts on the terms with a time rate in test functions, we acquire:

$$\int_\tau \int_{\mathcal{B}_0} \left(\frac{\partial \mathcal{L}}{\partial u_i} \delta u_i - \left(\frac{\partial \mathcal{L}}{\partial \dot{u}_i} \right)^{\cdot} \delta u_i + \frac{\partial \mathcal{L}}{\partial u_{i,j}} \delta u_{i,j} + \left(\frac{\partial \mathcal{L}}{\partial \ddot{u}_i} \right)^{\cdot\cdot} \delta u_i + \right. \\ \left. - \left(\frac{\partial \mathcal{L}}{\partial \dot{u}_{i,j}} \right)^{\cdot} \delta u_{i,j} + \frac{\partial \mathcal{L}}{\partial u_{i,jk}} \delta u_{i,jk} \right) \mathrm{d}V \, \mathrm{d}t = 0 \ . \tag{34}$$

The boundary terms in time vanish by selecting the time domain in a specific way. The same arguments are used for a differential equation in time; instead of using an initial and an end value, we can use an initial value and a value for its rate. Furthermore, we may write the integral form locally in time:

$$\int_{\mathcal{B}_0} \left(\frac{\partial \mathcal{L}}{\partial u_i} \delta u_i - \left(\frac{\partial \mathcal{L}}{\partial \dot{u}_i} \right)^{\cdot} \delta u_i + \frac{\partial \mathcal{L}}{\partial u_{i,j}} \delta u_{i,j} + \left(\frac{\partial \mathcal{L}}{\partial \ddot{u}_i} \right)^{\cdot\cdot} \delta u_i - \right. \\ \left. - \left(\frac{\partial \mathcal{L}}{\partial \dot{u}_{i,j}} \right)^{\cdot} \delta u_{i,j} + \frac{\partial \mathcal{L}}{\partial u_{i,jk}} \delta u_{i,jk} \right) \mathrm{d}V = 0 \ . \tag{35}$$

Moreover, for the boundary values we can select an appropriate action:

$$\mathcal{S} = \int_\tau \int_{\mathcal{B}_0} \mathcal{L} \, \mathrm{d}V \, \mathrm{d}t + \int_\tau \int_{\partial \mathcal{B}_0} W_\mathrm{s} \, \mathrm{d}A \, \mathrm{d}t + \int_\tau \int_{\partial \partial \mathcal{B}_0} W_\mathrm{e} \, \mathrm{d}\ell \, \mathrm{d}t \ , \tag{36}$$

where on the boundary surface a potential, W_s, and on the boundary edge another potential, W_e, have to be prescribed. After the analogous steps, the integral form becomes:

$$\int_{\mathcal{B}_0} \left(\frac{\partial \mathcal{L}}{\partial u_i} \delta u_i - \left(\frac{\partial \mathcal{L}}{\partial \dot{u}_i} \right)^{\cdot} \delta u_i + \frac{\partial \mathcal{L}}{\partial u_{i,j}} \delta u_{i,j} + \left(\frac{\partial \mathcal{L}}{\partial \ddot{u}_i} \right)^{\cdot\cdot} \delta u_i - \left(\frac{\partial \mathcal{L}}{\partial \dot{u}_{i,j}} \right)^{\cdot} \delta u_{i,j} + \right. \\ \left. + \frac{\partial \mathcal{L}}{\partial u_{i,jk}} \delta u_{i,jk} \right) \mathrm{d}V + \int_{\partial \mathcal{B}_0} \frac{\partial W_\mathrm{s}}{\partial u_i} \delta u_i \, \mathrm{d}A + \int_{\partial \partial \mathcal{B}_0} \frac{\partial W_\mathrm{e}}{\partial u_{i,j}} \delta u_{i,j} \, \mathrm{d}\ell = 0 \ . \tag{37}$$

For elasticity we choose the following LAGRANGEan density and boundary potentials:

$$\mathcal{L} = \frac{1}{2} \rho_0 \dot{u}_i \dot{u}_i - \rho_0 \psi + \rho_0 (f_i u_i + l_{ij} u_{i,j}) \ , \quad W_\mathrm{s} = \hat{t}_i u_i + \hat{m}_{ij} u_{j,i} \ , \quad W_\mathrm{e} = 0 \ , \tag{38}$$

where the free energy density, $\rho_0 \psi$, is simply the stored energy density, w, in the case of isothermal systems. The stored energy density is a potential such that it depends on $u_{i,j}$ and $u_{i,jk}$, but not on their time rates. The first term is the kinetic energy

density and the last terms model the supply terms performed on the displacement and its gradient. The traction vector, $\hat{t}_i$, is given as a force per unit area and by the torsion vector, $\hat{m}_{ij}$, a moment per unit area is applied. Moreover, we neglect an edge boundary energy by setting $W_\mathrm{e} = 0$ for easier comparison with the weak form attained from rational mechanics. Finally, by inserting the LAGRANGEan density and boundary potentials into the integral form we obtain:

$$\begin{aligned}\int_{\mathcal{B}_0} \Big(\rho_0 f_i \delta u_i - \rho_0 \ddot{u}_i \delta u_i + \Big(l_{ij} - \frac{\partial w}{\partial u_{i,j}} \Big) \delta u_{i,j} - \frac{\partial w}{\partial u_{i,jk}} \delta u_{i,jk} \Big) \, \mathrm{d}V + \\ + \int_{\partial \mathcal{B}_0} \big(\hat{t}_i \delta u_i + \hat{m}_{ij} \delta u_{j,i} \big) \, \mathrm{d}A = 0 \ .\end{aligned} \tag{39}$$

Since it is zero we can multiply it by minus one. We discretize in time by using the same scheme as before and acquire the weak form:

$$\begin{aligned}\mathrm{Form} = \int_{\mathcal{B}_0} \Big(- \rho_0 f_i \delta u_i + \rho_0 \frac{u_i - 2u_i^0 + u_i^{00}}{\Delta t^2} \delta u_i - \Big(\rho_0 l_{ij} - \frac{\partial w}{\partial u_{i,j}} \Big) \delta u_{i,j} + \\ + \frac{\partial w}{\partial u_{i,jk}} \delta u_{i,jk} \Big) \, \mathrm{d}V - \int_{\partial \mathcal{B}_0} \big(\hat{t}_i \delta u_i + \hat{m}_{ij} \delta u_{j,i} \big) \, \mathrm{d}A = 0 \ .\end{aligned} \tag{40}$$

Out of that weak form we can compute the displacement, $u_i(X_j, t)$, by using the finite element method. We obtain the same displacement field if the weak forms obtained by the rational and classical analytical mechanics are identical.

4 Comparison and discussion

We have formulated one weak form of generalized elasticity by using rational mechanics in Eq. (23) and another weak form of generalized elasticity by using classical analytical mechanics in Eq. (40). A comparison shows an interesting result. We start with:

$$\frac{\partial w}{\partial u_{i,j}} = \frac{\partial w}{\partial E_{kl}} \frac{\partial E_{kl}}{\partial u_{i,j}} = S_{lk} F_{nk} \frac{\partial F_{nl}}{\partial u_{i,j}} = S_{lk} F_{nk} \frac{\partial u_{n,l}}{\partial u_{i,j}} = S_{lk} F_{nk} \delta_{ni} \delta_{lj} = S_{jk} F_{ik} \ , \tag{41}$$

and obtain after using the relation from Eq. $(15)_1$, $F_{kj} S_{ij} = P_{ik}$,

$$\frac{\partial w}{\partial u_{i,j}} = P_{ji} \ . \tag{42}$$

The latter shows that these two terms are equal in the weak forms. However, this is not the case for the following term:

$$\begin{aligned}\frac{\partial w}{\partial u_{i,jk}} = \frac{\partial w}{\partial E_{lm,n}} \frac{\partial E_{lm,n}}{\partial u_{i,jk}} = \mu_{nml} \frac{\partial F_{ol} F_{om,n}}{\partial u_{i,jk}} = \mu_{nml} F_{ol} \frac{\partial u_{o,mn}}{\partial u_{i,jk}} = \\ = \mu_{nml} F_{ol} \delta_{oi} \delta_{mj} \delta_{nk} = \mu_{kjl} F_{il} \ .\end{aligned} \tag{43}$$

This latter inconsistency may be amended by suggesting another formulation of the stored energy. Moreover, in Eq. (23) there is one term, $P_{[ij]}$, which does not vanish in

general. This term is introduced by first transforming into the LAGRANGEan frame and then defining the angular momentum with respect to X_i. This fact is another inconsistency between the formulations. These two ambiguities can be eliminated only for a special case, where $F_{ij} \approx \delta_{ij}$ is assumed. Then $P_{ij} \approx \sigma_{ij}$ and we know that for non-polar materials $\sigma_{ij} = \sigma_{ji}$ leading to $\sigma_{[ij]} = 0$. This case is known as neglecting the geometric nonlinearities so that the formulation is accurate only for small deformations.

By using two different methodologies we have realized that there are some ambiguities easily introduced to the formulation by both of the methodologies. Either we postulate a stored energy bringing an inconsistency, or transform from the current into the LAGRANGEan frame and define the physical variable in a specific way introducing another inconsistency. The correct formulation can be suggested by learning from both of methodologies. Unfortunately, many researchers get used to only one of the methodologies and, even further, they pretend that their used methodology is the *only true one*. We believe that human-beings' creativity is great enough to simply incorporate some ambiguities into one formulation. Hence, two different possibilities leading to the same formulation is an invaluable double-check for us. We have achieved a generalized elasticity only for small deformations and left a better theory including geometric nonlinearities to further research.

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