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ORIGINAL ARTICLE

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Revealing the physical insight of a length-scale parameter in metamaterials by exploiting the variational formulation

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Abstract For micro-architectured materials with a substructure, called metamaterials, we can realize a direct numerical simulation in the microscale by using classical mechanics. This method is accurate, however, computationally costly. Instead, a solution of the same problem in the macroscale is possible by means of the generalized mechanics. In this case, no detailed modeling of the substructure is necessary; however, new parameters emerge. A physical interpretation of these metamaterial parameters is challenging leading to a lack of experimental strategies for their determination. In this work, we exploit the variational formulation based on action principles and obtain a direct relation between a parameter used in the kinetic energy and a metamaterial parameter in the case of a viscoelastic model.

Keywords Mechanics · Metamaterial · Length scale parameter · Variational formulation · Principle of least action

1 Introduction

Developments of the technology in manufacturing, especially the additive methods called 3D printing, allow us to construct metamaterials, which are materials with a substructure in the microscale. This geometric substructure is not detected on the macroscale; however, it affects the response of the metamaterial. Hence, a functionalized material is possible to design for specific purposes. Moreover, there exist many materials like concrete, polymers, or even aluminum, where the composition of the material (porosity, chains, grains) becomes dominating in the case of small geometric sizes. This phenomenon is called the size-effect and can be successfully described by means of the generalized mechanics.

The idea and applications of generalized mechanics are older than a century, see dell'Isola et al. [12]; but the modern theory started with Mindlin and Tiersten [33], Mindlin [32], Toupin [54], Eringen [16]. Today, there are different methodologies generalizing mechanics; we can comprehend them under the hood of a unified theory, see Neff et al. [35]. We know from the experiments conducted in Morrison [34], Yang and Lakes [56], Brezny and Green [8], Chen and Fleck [9], Kesler and Gibson [24], Jing et al. [23], Chen et al. [10], Liebold and Müller [28] that a second-order theory is suitable for obtaining an accurate prediction of such experiments. Different computational approaches implementing second-order theories have been suggested in the literature, for a recent review see Thai et al. [53]. Some authors suggest using special elements as in Bilotta et al. [7], which is compared between different scales as in Pideri and Seppecher [40] and Giorgio [18]. Some authors use nonlocal solution strategies as in Niiranen et al. [37]. A similar approach is called peridynamics, also known

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for a long time, see dell'Isola et al. [13]; this method is getting more attention recently, see Silling [48], Silling and Lehoucq [49], Oterkus and Madenci [39], Diyaroglu et al. [15], Taylor and Steigmann [52], De Meo et al. [11], Queiruga and Moridis [45]. For some interesting engineering examples by using generalized mechanics, we refer to Scerrato et al. [47], Turco et al. [55], and Placidi et al. [41]. A finite element implementation can be found in Reiher et al. [46] as well as in Abali et al. [3].

The theory of mechanics is formulated by an expression called LAGRANGEan that depends on primitive variables and their *first* derivatives in coordinates. The generalization is possible by adding the *second* derivatives in coordinates to the argument list of this LAGRANGEan. By following this strategy, we obtain a second-order theory and generate a weak form for isothermal and elastic bodies. Technically, a weak form is the only necessary piece of information for a successful computational implementation; see Abali [2] for various engineering examples and their implementations by using the weak form.

In this work, we briefly present the strategy for generating a second-order theory and its weak form by means of action principles. The only postulate is an existing LAGRANGEan density modeling the system accurately. This method is often used for reversible systems; however, it is quite straightforward to introduce a dissipation function such that we obtain an irreversible system by means of the same strategy. We use displacement as the primitive variable and obtain a variational form for isothermal mechanics as well as generalized mechanics. Therefore, herein, a reversible system is simply a system composed of an elastic material, and an irreversible system possesses a viscoelastic material. We exploit the action principle in a standard and well-known manner, so the key concept is the dissipation function and its definition. There are various strategies for obtaining the dissipation function in the literature, for example, in Gurtin and Reddy [20], Polizzotto [42], Lubarda [30], different starting points have been chosen. We start off with discussing the first- and second-order theories in a general way, and then we use the machinery to gain the weak form. We manage to introduce the concept in a simple yet sophisticated manner allowing us to generalize for elastic and viscoelastic metamaterials. Additionally, we gain an understanding of a metamaterial parameter introduced in the kinetic energy as being tantamount to the viscoelastic metamaterial parameter. This physical insight is of importance in order to figure out a possible experimental strategy for determining this parameter.

2 First- and second-order theories

The well-known analytical mechanics as in Lanczos [26] is based on the axiom that we can define a LAGRANGEan density:

$$\mathcal{L} = \mathcal{L}\big(x^\mu, \phi_A, \phi_{A,\mu}\big), \tag{1}$$

which successfully describes the underlying system. This function depends on coordinates in m-dimensional space $\Omega \subset \mathbb{R}^m$ with $\mu = 1, 2, \dots m$, primitive variables, and their derivatives,

$$x^\mu \in \Omega, \quad \phi_A = \phi_A(x^\mu), \quad \phi_{A,\mu} = \frac{\partial \phi_A}{\partial x^\mu}, \tag{2}$$

respectively. The topology of x^μ is restricted to be represented by oblique coordinates—CHRISTOFFEL symbols vanish—with a metric determinant of 1 such that we introduce the action

$$\mathcal{A} = \int_\Omega \mathcal{L}\,\mathrm{d}\Sigma + \int_{\partial\Omega} W_\mathrm{s}\,\mathrm{d}\Gamma, \quad \mathrm{d}\Sigma = \mathrm{d}x^1\,\mathrm{d}x^2 \cdots \mathrm{d}x^m, \quad \mathrm{d}\Gamma = \mathrm{d}x^1\,\mathrm{d}x^2 \cdots \mathrm{d}x^{m-1}, \tag{3}$$

with a volume element $\mathrm{d}\Sigma$ and a surface element $\mathrm{d}\Gamma$, which is one dimension less than the volume element. This abstract formalism is indeed very fruitful as we will present in the following. It is based on the invariance properties of the action as developed in Noether [38] for analytic fields, i.e., the primitive variables are smooth functions in x^μ. We will use space-time for x^μ, displacement for ϕ_A in order to describe isothermal mechanics.

The set of arguments of the LAGRANGEan density $\{x^\mu, \phi_A, \phi_{A,\mu}\}$ is decomposed to the coordinates x^μ (independent variables) and the so-called phase space $\{\phi_A, \phi_{A,\mu}\}$ depending on coordinates. Under a transformation of the phase space, the action remains invariant for a reversible process. In the case of an irreversible process, the transformation produces a dissipation given as a *virtual* power $\delta\mathcal{D}$ in the sense that this transformation is an arbitrary and virtual process given by

$$\mathcal{D} = \int_\Omega \sigma\,\mathrm{d}\Sigma, \quad \sigma = \sigma\big(x^\mu, \phi_A, \phi_{A,\mu}\big). \tag{4}$$

Hence, a mathematically equivalent approach is called the method of virtual power based on D'ALEMBERT's principle in rigid body mechanics with the independent variable x^μ being the time. For a deformable body, the method is called the principle of virtual power or the principle of least action; we refer to Neuenschwander [36] for a thorough discussion of the action principle. In order to calculate the aforementioned transformation of the phase space, we apply

$$\begin{aligned} \phi'_A &= \phi_A + \varepsilon\delta\phi_A, \\ \phi'_{A,\mu} &= \phi_{A,\mu} + \varepsilon\delta\phi_{A,\mu}, \end{aligned} \tag{5}$$

with a constant ε and arbitrary $\delta\phi_A$, $\delta\phi_{A,\mu}$. We are interested in a solution *within* the boundary Ω, and the solution *on* the boundary is known such that we can choose $\delta\phi_A$ and $\delta\phi_{A,\mu}$ zero on boundaries. By introducing ε, we can determine the action principle:

$$\delta\mathcal{A} = \delta\mathcal{D}, \tag{6}$$

with

$$\begin{aligned} \delta\mathcal{A} &= \left(\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\int_\Omega \mathcal{L}\big(x^\mu, \phi'_A, \phi'_{A,\mu}\big)\,\mathrm{d}\Sigma\right)\bigg|_{\varepsilon=0} + \left(\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\int_{\partial\Omega} W_\mathrm{s}\big(x^\mu, \phi'_A\big)\,\mathrm{d}\Gamma\right)\bigg|_{\varepsilon=0}, \\ \delta\mathcal{D} &= \left(\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\int_\Omega \sigma\big(x^\mu, \phi'_A, \phi'_{A,\mu}\big)\,\mathrm{d}\Sigma\right)\bigg|_{\varepsilon=0}, \end{aligned} \tag{7}$$

which is indeed possible to rewrite as $\delta\bar{\mathcal{A}} = 0$ by introducing $\bar{\mathcal{A}} = \mathcal{A} - \mathcal{D}$. Possible transformations are called canonical transformations, and all canonical transformations leaving the action unchanged form a group. The description in Eq. (7) generates an equation without ε as follows

$$\int_\Omega \left(\frac{\partial(\mathcal{L}-\sigma)}{\partial\phi_A}\delta\phi_A + \frac{\partial(\mathcal{L}-\sigma)}{\partial\phi_{A,\mu}}\delta\phi_{A,\mu}\right)\mathrm{d}\Sigma + \int_{\partial\Omega}\frac{\partial W_\mathrm{s}}{\partial\phi_A}\delta\phi_A\,\mathrm{d}\Gamma = 0, \tag{8}$$

where and henceforth over repeated indices we apply the EINSTEIN summation convention. At this stage, an integration by parts on all coordinates is established for deriving the so-called EULER–LAGRANGE equations. We will see in the coming sections that the above formalism leads directly to the weak form without integrating by parts on all coordinates.

The same strategy is used to propose the generalization to the second-order theory; this time, the starting postulate is renewed by adding the second derivative related to coordinates such that the LAGRANGEan density and the dissipation function read

$$\mathcal{L} = \mathcal{L}\big(x^\mu, \phi_A, \phi_{A,\mu}, \phi_{A,\mu\nu}\big), \quad \sigma = \sigma\big(x^\mu, \phi_A, \phi_{A,\mu}, \phi_{A,\mu\nu}\big). \tag{9}$$

Since we have introduced two derivatives, the action reads

$$\begin{aligned} \mathcal{A} &= \int_\Omega \mathcal{L}\,\mathrm{d}\Sigma + \int_{\partial\Omega} W_\mathrm{s}\,\mathrm{d}\Gamma + \int_{\partial\Omega} W_\mathrm{e}\,\mathrm{d}\Pi, \\ \mathrm{d}\Sigma &= \mathrm{d}x^1\,\mathrm{d}x^2\cdots\mathrm{d}x^m, \quad \mathrm{d}\Gamma = \mathrm{d}x^1\,\mathrm{d}x^2\cdots\mathrm{d}x^{m-1}, \\ \mathrm{d}\Gamma &= \mathrm{d}x^1\,\mathrm{d}x^2\cdots\mathrm{d}x^{m-2}, \end{aligned} \tag{10}$$

where the work done on surface W_s and the work done on edge W_e depend on ϕ_A and $\phi_{A,\mu}$. Surface and edge mean for one and two dimensions lower boundaries, respectively. By using the same transformation for the phase space $\{\phi_A, \phi_{A,\mu}, \phi_{A,\mu\nu}\}$ with the constant ε and arbitrary $\delta\phi_A$, $\delta\phi_{A,\mu}$, $\delta\phi_{A,\mu\nu}$ vanishing on the boundary, we obtain

$$\begin{aligned} &\int_\Omega \left(\frac{\partial(\mathcal{L}-\sigma)}{\partial\phi_A}\delta\phi_A + \frac{\partial(\mathcal{L}-\sigma)}{\partial\phi_{A,\mu}}\delta\phi_{A,\mu} + \frac{\partial(\mathcal{L}-\sigma)}{\partial\phi_{A,\mu\nu}}\delta\phi_{A,\mu\nu}\right)\mathrm{d}\Sigma \\ &+ \int_{\partial\Omega}\left(\frac{\partial W_\mathrm{s}}{\partial\phi_A}\delta\phi_A + \frac{\partial W_\mathrm{s}}{\partial\phi_{A,\mu}}\delta\phi_{A,\mu}\right)\mathrm{d}\Gamma \\ &+ \int_{\partial\partial\Omega}\left(\frac{\partial W_\mathrm{e}}{\partial\phi_A}\delta\phi_A + \frac{\partial W_\mathrm{e}}{\partial\phi_{A,\mu}}\delta\phi_{A,\mu}\right)\mathrm{d}\Pi = 0, \end{aligned} \tag{11}$$

which is the integral form of the second-order theory analog to Eq. (8) deduced for the first-order theory.

3 Generating the weak form

Equations (8), (11) will be applied for a material system. Consider a three-dimensional continuum body in a reference frame, $\mathcal{B}_0 \subset \mathbb{R}^3$, with its closure, $\partial\mathcal{B}_0$, such that $\bar{\mathcal{B}_0} = \mathcal{B}_0 \cup \partial\mathcal{B}_0$. Material particles of the continuum body are denoted by their positions, $\boldsymbol{X}$, in the reference frame. We can choose the initial frame as the reference frame indicated by use of an index "0." This choice is indeed practical in a simulation, which begins with the known displacements, u_i, initially. The displacement field is the primitive variable $\phi_A = u_i$ depending on time t and space X_i expressed in Cartesian coordinates, i.e., $x^\mu = (t, X_i)$.

We start with the first-order theory, and after inserting the coordinates as well as the primitive variables into Eq. (8), we obtain

$$\int_\tau \int_{\mathcal{B}_0} \left(\frac{\partial(\mathcal{L}-\sigma)}{\partial u_i}\delta u_i + \frac{\partial(\mathcal{L}-\sigma)}{\partial u_i^\cdot}\delta u_i^\cdot + \frac{\partial(\mathcal{L}-\sigma)}{\partial u_{i,j}}\delta u_{i,j} \right) \mathrm{d}t\,\mathrm{d}V + \int_\tau \int_{\partial\mathcal{B}_0} \frac{\partial W_\mathrm{s}}{\partial u_i}\delta u_i\,\mathrm{d}t\,\mathrm{d}A = 0, \tag{12}$$

where $(\cdot)^\cdot$ denotes a material rate being identical with the partial time derivative since the initial frame is fixed in time. After discretizing in space and time, we will fulfill this integral form. In order to discretize in time, we apply the EULER backwards method:

$$\boldsymbol{u}^\cdot = \frac{\partial \boldsymbol{u}}{\partial t} = \frac{\boldsymbol{u} - \boldsymbol{u}^0}{\Delta t}, \tag{13}$$

where $\boldsymbol{u}^0$ represents the value from the last time step. For the space discretization, we apply the finite element method by skipping an explicit notational difference between analytic functions and their numerical approximations. The so-called test functions, $\delta\boldsymbol{u}$, are arbitrary but from the same space as the displacements, $\boldsymbol{u}$, known as GALERKIN's approach. For the first-order theory, we can use linear finite elements, $n = 1$, belonging to the following SOBOLEV space

$$\mathcal{V} = \left\{ \boldsymbol{u} \in [\mathcal{H}^n(\Omega)]^3 : \boldsymbol{u}|_{\partial\Omega} = \text{given} \right\}, \tag{14}$$

where $[\mathcal{H}^n]^3$ is a three-dimensional HILBERT space as in Hilbert [21] with the differentiability properties. We choose the test functions from the same space and as constants in time such that we need to eliminate next the rates from the integral form by integrating by parts and using given values at two subsequent time instants. Thus, we obtain

$$\int_{\mathcal{B}_0} \left(\frac{\partial(\mathcal{L}-\sigma)}{\partial u_i}\delta u_i - \left(\frac{\partial(\mathcal{L}-\sigma)}{\partial u_i^\cdot}\right)^\cdot \delta u_i + \frac{\partial(\mathcal{L}-\sigma)}{\partial u_{i,j}}\delta u_{i,j} \right) \mathrm{d}V + \int_{\partial\mathcal{B}_0} \frac{\partial W_\mathrm{s}}{\partial u_i}\delta u_i\,\mathrm{d}A = 0. \tag{15}$$

Another integration by parts is used for exchanging the space derivative in the test functions, and the outcome is called the EULER–LAGRANGE equations leading directly to the balance equations where the regularity condition is stronger than the latter integral form. Therefore, we call Eq. (15) a *weak* form in each finite element, E. Its summation over all domain reads

$$\text{Form} = \sum_{n=1}^{\text{nr of ele.}} \int_{E^n} \left(\frac{\partial(\mathcal{L}-\sigma)}{\partial u_i}\delta u_i - \left(\frac{\partial(\mathcal{L}-\sigma)}{\partial u_i^\cdot}\right)^\cdot \delta u_i + \frac{\partial(\mathcal{L}-\sigma)}{\partial u_{i,j}}\delta u_{i,j} \right) \mathrm{d}V + \int_{\partial\mathcal{B}_0} \frac{\partial W_\mathrm{s}}{\partial u_i}\delta u_i\,\mathrm{d}A. \tag{16}$$

The external work done on the surface, $W_\mathrm{s} = W_\mathrm{s}(t, X_i, \phi_A)$, is a given function.

In the case of the second-order theory, we use similar steps and obtain from Eq. (11) the following integral form for the displacement as being the sole primitive variable

$$\begin{aligned}
&\int_\Omega \Big(\frac{\partial(\mathcal{L}-\sigma)}{\partial u_i}\delta u_i + \frac{\partial(\mathcal{L}-\sigma)}{\partial \dot{u}_i}\delta \dot{u}_i + \frac{\partial(\mathcal{L}-\sigma)}{\partial u_{i,j}}\delta u_{i,j} + \frac{\partial(\mathcal{L}-\sigma)}{\partial \ddot{u}_i}\delta \ddot{u}_i \\
&\quad + 2\frac{\partial(\mathcal{L}-\sigma)}{\partial \dot{u}_{i,j}}\delta \dot{u}_{i,j} + \frac{\partial(\mathcal{L}-\sigma)}{\partial u_{i,jk}}\delta u_{i,jk}\Big)\,\mathrm{d}\Sigma \\
&\quad + \int_{\partial\Omega}\Big(\frac{\partial W_\mathrm{s}}{\partial u_i}\delta u_i + \frac{\partial W_\mathrm{s}}{\partial \dot{u}_i}\delta \dot{u}_i + \frac{\partial W_\mathrm{s}}{\partial u_{i,j}}\delta u_{i,j}\Big)\,\mathrm{d}\Gamma \\
&\quad + \int_{\partial\partial\Omega}\Big(\frac{\partial W_\mathrm{e}}{\partial u_i}\delta u_i + \frac{\partial W_\mathrm{e}}{\partial \dot{u}_i}\delta \dot{u}_i + \frac{\partial W_\mathrm{e}}{\partial u_{i,j}}\delta u_{i,j}\Big)\,\mathrm{d}\Pi = 0.
\end{aligned} \tag{17}$$

Analogously, we eliminate the rate of test functions by integrating by parts; after summing over all elements composing the computational domain, we obtain the weak form for the second-order theory

$$\begin{aligned}
\text{Form} = \sum_{n=1}^{\text{nr of ele.}} &\int_{E^n} \Big(\frac{\partial(\mathcal{L}-\sigma)}{\partial u_i}\delta u_i - \Big(\frac{\partial(\mathcal{L}-\sigma)}{\partial \dot{u}_i}\Big)^{\cdot}\delta u_i + \frac{\partial(\mathcal{L}-\sigma)}{\partial u_{i,j}}\delta u_{i,j} + \Big(\frac{\partial(\mathcal{L}-\sigma)}{\partial \ddot{u}_i}\Big)^{\cdot\cdot}\delta u_i \\
&- 2\Big(\frac{\partial(\mathcal{L}-\sigma)}{\partial \dot{u}_{i,j}}\Big)^{\cdot}\delta u_{i,j} + \frac{\partial(\mathcal{L}-\sigma)}{\partial u_{i,jk}}\delta u_{i,jk}\Big)\,\mathrm{d}\Sigma \\
&+ \int_{\partial\Omega}\Big(\frac{\partial W_\mathrm{s}}{\partial u_i}\delta u_i - \Big(\frac{\partial W_\mathrm{s}}{\partial \dot{u}_i}\Big)^{\cdot}\delta u_i + \frac{\partial W_\mathrm{s}}{\partial u_{i,j}}\delta u_{i,j}\Big)\,\mathrm{d}\Gamma \\
&+ \int_{\partial\partial\Omega}\Big(\frac{\partial W_\mathrm{e}}{\partial u_i}\delta u_i - \Big(\frac{\partial W_\mathrm{e}}{\partial \dot{u}_i}\Big)^{\cdot}\delta u_i + \frac{\partial W_\mathrm{e}}{\partial u_{i,j}}\delta u_{i,j}\Big)\,\mathrm{d}\Pi.
\end{aligned} \tag{18}$$

Obviously, we need to use quadratic elements, $n = 2$, for the space defined in Eq. (14) in order to represent twice differentiations in space occurring in the weak form.

Both weak forms in Eqs. (16), (18) are general in the sense that they are valid for any geometry and material as long as the assumption of an isothermal process is accurate enough. In order to make the weak form useful, we need to define the scalar functions $\mathcal{L}$ and σ modeling the underlying material.

3.1 Elasticity in the first-order theory

Consider an elastic material with the following scalar functions

$$\mathcal{L} = \frac{1}{2}\rho_0 \dot{u}_i \dot{u}_i - w + \rho_0 f_i u_i, \quad \sigma = 0, \tag{19}$$

as well as the stored energy density

$$w = \frac{1}{2}\varepsilon_{ij} C_{ijkl} \varepsilon_{kl}, \tag{20}$$

which is quadratic in strain $\boldsymbol{\varepsilon}$ with the constant stiffness tensor $\boldsymbol{C}$ known as linear elasticity. The specific force f_i is the known body force due to the gravitation. The mass density ρ_0 is a known function in $\boldsymbol{X}$ for heterogeneous or a known constant value for homogeneous materials. By inserting Eqs. (19), (20) into the form in Eq. (16), we obtain the necessary weak form for a computation

$$\text{Form} = \sum_{n=1}^{\text{nr of ele.}} \int_{E^n} \Big(\rho_0 f_i \delta u_i - \rho_0 \ddot{u}_i \delta u_i - \frac{\partial w}{\partial u_{i,j}}\delta u_{i,j}\Big)\,\mathrm{d}V + \int_{\partial\mathcal{B}_0} \frac{\partial W_\mathrm{s}}{\partial u_i}\delta u_i\,\mathrm{d}A. \tag{21}$$

The work done on the surface is often given by a traction vector, $\hat{t}_i$, in N/m^2

$$W_\mathrm{s} = \hat{t}_i u_i. \tag{22}$$

For conceptual simplicity, we neglect the geometric nonlinearities and use a linear strain measure:

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \tag{23}$$

leading to the multiplier of $-\delta u_{i,j}$ as follows:

$$\frac{\partial w}{\partial u_{i,j}} = \frac{\partial w}{\partial \varepsilon_{kl}}\frac{\partial \varepsilon_{kl}}{\partial u_{i,j}} = \frac{\partial w}{\partial \varepsilon_{kl}}\frac{1}{2}(\delta_{ki}\delta_{lj} + \delta_{kj}\delta_{li}) = \frac{\partial w}{\partial \varepsilon_{ij}} = C_{ijkl}\varepsilon_{kl}, \tag{24}$$

since the strain tensor is symmetric. The latter is called stress; its definition is given by the stored energy density. By choosing a nonlinear stored energy, any hyperelastic material model can be implemented without difficulties; the weak form obtained in Eq. (21) is general.

3.2 Viscoelasticity in the first-order theory

In order to include the material damping because of the viscous character, we use the same LAGRANGEan density but a nonzero dissipation function

$$\mathcal{L} = \frac{1}{2}\rho_0 \dot{u}_i \dot{u}_i - w + \rho_0 f_i u_i, \quad \sigma = \varepsilon_{ij} A_{ijkl} \dot{\varepsilon}_{kl}, \tag{25}$$

leading to the following weak form:

$$\text{Form} = \sum_{n=1}^{\text{nr of ele.}} \int_{E^n} \left(\rho_0 f_i \delta u_i - \rho_0 \ddot{u}_i \delta u_i - \frac{\partial (w+\sigma)}{\partial u_{i,j}} \delta u_{i,j}\right) \mathrm{d}V + \int_{\partial \mathcal{B}_0} \hat{t}_i \delta u_i \, \mathrm{d}A, \tag{26}$$

by means of the surface work as in Eq. (22). Again for simplicity, let us use the quadratic stored energy function as in Eq. (20) and figure out the so-called stress

$$-\frac{\partial (\mathcal{L} - \sigma)}{\partial u_{i,j}} = \frac{\partial (w+\sigma)}{\partial u_{i,j}} = C_{ijkl}\varepsilon_{kl} + A_{ijkl}\dot{\varepsilon}_{kl}, \tag{27}$$

which is the anisotropic form of the KELVIN material—for determination of parameters in viscoelastic materials, we refer to Flügge [17] in case of linear models and Abali et al. [5] for nonlinear models. Again, the weak form in Eq. (26) is general and can be applied for any type of viscoelastic material model.

It is of importance to emphasize that σ and w could be merged into one scalar function and its dependency on strain and strain rate would be postulated as a starting point. This strategy is mathematically admissible, but a straightforward consequence of this approach leads to a quadratic term in strain rate, which indeed vanishes in the first-order theory. Hence, we propose to consider stored energy density, w, and dissipation density, σ, as distinct functions. Their only difference is that the stored energy density depends on strain and thus recoverable; however, the dissipation density depends on strain and its rate such that σ introduces the irreversibility into the system. Even if σ possesses a strain rate quadratic term, the first-order theory fails to capture any difference in the deformation. We need an extension of the theory to the second order as a possible amendment.

3.3 Elasticity in the second-order theory

For a better comparison between the first- and second-order theories, consider the same LAGRANGEan density and surface work in the case of elasticity

$$\mathcal{L} = \frac{1}{2}\rho_0 \dot{u}_i \dot{u}_i - w + \rho_0 f_i u_i, \quad \sigma = 0, \quad W_\mathrm{s} = \hat{t}_i u_i, \quad W_\mathrm{e} = 0, \tag{28}$$

which generates from Eq. (18) the following weak form:

$$\text{Form} = \sum_{n=1}^{\text{nr of ele.}} \int_{E^n} \left(\rho_0 f_i \delta u_i - \rho_0 \ddot{u}_i \delta u_i - \frac{\partial w}{\partial u_{i,j}} \delta u_{i,j} - \frac{\partial w}{\partial u_{i,jk}} \delta u_{i,jk}\right) \mathrm{d}\Sigma + \int_{\partial \Omega} \hat{t}_i \delta u_i \, \mathrm{d}\Gamma, \tag{29}$$

since the stored energy density fails to depend on rate of strain. This weak form has an additional term characterizing the materials known from the strain gradient theories. The stored energy density depends on strain and its gradient (space derivative). As investigated in dell'Isola et al. [14], the simplest representation of such a term reads

$$w = \frac{1}{2}\varepsilon_{ij}C_{ijkl}\varepsilon_{kl} + \varepsilon_{ij}G_{ijklm}\varepsilon_{kl,m} + \varepsilon_{ij,k}D_{ijklmn}\varepsilon_{lm,n}, \tag{30}$$

where all material coefficients $\boldsymbol{C}$, $\boldsymbol{G}$, $\boldsymbol{D}$ depend on $\boldsymbol{X}$ for a heterogeneous material; they are constant in the case of a homogeneous material. The mixed term vanishes, $\boldsymbol{G} = 0$, for a centrosymmetric crystal structure, for example, materials showing a piezoelectric property are non-centrosymmetric. For the isotropic case, we refer to Suiker and Chang [51], Abali et al. [4, Appendix], Lazar [27] for the representation of material coefficients. Even in the case of isotropic strain gradient materials, the additional coefficient $\boldsymbol{D}$ introduces 5 new constants to be determined by means of experiments. There are various attempts to determine some of these constants, as in Chen and Fleck [9], Kesler and Gibson [24], Lam et al. [25], McFarland and Colton [31], Gruber et al. [19], Liebold and Müller [29]; but no experimental strategy is known for determining all of these 5 constants.

Moreover, the second-order theory is capable of capturing more complicated inertial as well as supply terms. Consider the following LAGRANGEan density

$$\mathcal{L} = \frac{1}{2}\rho_0\left(\dot{u}_i\dot{u}_i + d^2\dot{u}_{i,j}\dot{u}_{i,j} + \tau^2\ddot{u}_i\ddot{u}_i\right) - w + \rho_0\left(f_i u_i + \ell_{ij}u_{i,j}\right). \tag{31}$$

The inertial term possesses a length scale d in m—as suggested already in Mindlin [32], also see Polizzotto [43], Polizzotto [44] for this term—and a relaxation time τ in s. The supply term, ℓ_{ij}, allows a volumetric work affecting the displacement gradient directly. Although theoretically possible, their role in a process is not yet fully understood. We can even construct surface and edge works on boundary

$$W_\mathrm{s} = \hat{t}_i u_i + \hat{g}_i\dot{u}_i + \hat{k}_{ij}u_{i,j}, \quad W_\mathrm{e} = \hat{r}_i u_i + \hat{p}_i\dot{u}_i + \hat{s}_{ij}u_{i,j}. \tag{32}$$

Obviously, measurement or even a physical explanation of such terms is very challenging. We refer to discussions in Auffray et al. [6], Javili et al. [22], Steigmann and dell'Isola [50] for possible interpretations of some of these boundary and edge terms. We emphasize that the weak form in Eq. (18) generates the necessary form for a computation for all possible material models. After inserting Eqs. (31), (32) into Eq. (18), the following weak form reads

$$\begin{aligned}\text{Form} = \sum_{n=1}^{\text{nr of ele.}} \int_{E^n} \Big(&\rho_0 f_i\delta u_i - \rho_0\ddot{u}_i\delta u_i + \rho_0\ell_{ij}\delta u_{i,j} - \frac{\partial w}{\partial u_{i,j}}\delta u_{i,j} \\ &+ \rho_0\tau^2\dddot{u}_i\delta u_i - 2\rho_0 d^2\ddot{u}_{i,j}\delta u_{i,j} - \frac{\partial w}{\partial u_{i,jk}}\delta u_{i,jk}\Big)\,\mathrm{d}\Sigma \\ &+ \int_{\partial\Omega}\left(\hat{t}_i\delta u_i - \hat{g}_i\dot{\delta u}_i + \hat{k}_{ij}\delta u_{i,j}\right)\mathrm{d}\Gamma \\ &+ \int_{\partial\partial\Omega}\left(\hat{r}_i\delta u_i - \hat{p}_i\dot{\delta u}_i + \hat{s}_{ij}\delta u_{i,j}\right)\mathrm{d}\Pi,\end{aligned} \tag{33}$$

where the surface and edge fields, $\hat{\boldsymbol{t}}$, $\hat{\boldsymbol{g}}$, $\hat{\boldsymbol{k}}$, $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{p}}$, $\hat{\boldsymbol{s}}$, as well as the volumetric supply terms, $\boldsymbol{f}$, $\boldsymbol{\ell}$, are given functions. Stored energy density, w, is defined in terms of the only unknown, the displacement field $\boldsymbol{u}$.

3.4 Viscoelasticity in the second-order theory

For the second-order theory, in the case of elasticity, we have exploited all terms in the general weak form in Eq. (18). The stored energy density includes as arguments only displacement gradients. In viscoelasticity, as we have presented in the preceding section, the dissipation function introduces the rate of displacement

gradients. For a clearer demonstration, consider a similar LAGRANGEan density from the preceding section augmented by the dissipation function quadratic in strain rate

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\rho_0\left(\dot{u_i}\dot{u_i} + d^2\dot{u}_{i,j}\dot{u}_{i,j} + \tau^2\ddot{u_i}\ddot{u_i}\right) - w + \rho_0 f_i u_i,\\ \sigma &= -\dot{\varepsilon}_{ij}A_{ijkl}\dot{\varepsilon}_{kl} - \frac{1}{2}\dot{\varepsilon}_{ij}B_{ijkl}\dot{\varepsilon}_{kl}.\end{aligned} \tag{34}$$

We figure out that the multiplier of $-\delta u_{i,j}$ in this configuration reads

$$-\frac{\partial(\mathcal{L}-\sigma)}{\partial u_{i,j}} + 2\left(\frac{\partial(\mathcal{L}-\sigma)}{\partial \dot{u}_{i,j}}\right)^{\cdot} = \frac{\partial w}{\partial u_{i,j}} + A_{ijkl}\dot{\varepsilon}_{kl} + 2\rho_0 d^2\ddot{u}_{i,j} + 2B_{ijkl}\ddot{\varepsilon}_{kl}, \tag{35}$$

with $A_{ijkl} = A_{klij}$. Obviously, coefficient B_{ijkl} in front of the second strain rate as well as the parameter d^2 in the kinetic energy share the same information. For presenting this issue more clearly, consider an isotropic material

$$\begin{aligned}B_{ijkl} &= b_1\delta_{ij}\delta_{kl} + b_2\delta_{ik}\delta_{jl} + b_3\delta_{il}\delta_{jk},\\ \dot{\varepsilon}_{ij}B_{ijkl}\dot{\varepsilon}_{kl} &= b_1\dot{\varepsilon}_{ii}\dot{\varepsilon}_{kk} + b_2\dot{\varepsilon}_{ij}\dot{\varepsilon}_{ij} + b_3\dot{\varepsilon}_{i,j}\dot{\varepsilon}_{j,i},\end{aligned} \tag{36}$$

where b_2 has the same role as $\rho_0 d^2$. Hence, we can omit to introduce d at all. The material parameter b_2 takes over its role in the deformation behavior. This interesting observation leads to the conclusion that the length-scale parameter d—probably proposed for the first time in Mindlin [32]—is in fact the reduced form caused by the material parameter $\boldsymbol{B}$. In other words, a material-related property causing dissipation can also be interpreted as an inertial term. Although they are mathematically equivalent, the physical interpretation is of importance, especially if one wants to design an experiment. For the inertial term, it is difficult to obtain experimental data. However, we can collect data in the case of dissipation.

By exploiting the variational formulation, we gain a precise understanding of the physical meaning of the length-scale parameter emerging in the kinetic energy as a consequence of the substructure. Therefore, we can also design an experimental approach to determine this parameter. We emphasize the meaning of w as well as σ in the formulation: They are material- and substructure-related scalar functions in the unit of energy peer volume. Consider a measurement device, for example a rheometer, where we deform the material by using a harmonic function. Every cyclic deformation generates a stored energy as well as a dissipated energy. These energies can be precisely measured by recording the (effectively) used energy by the device, see Abali [1, Sect. 8]. Under the assumption of negligible inertial effects, we can correlate these measurable energies to w and σ. For different amplitudes and frequencies, several measurements generate enough data to determine the parameters by using an inverse analysis. At least for the parameter b_2, this method is straightforward. In the case of the metamaterial parameters, a rheometer is not a feasible method of collecting data. The rheometer has a thin layer of material such that the strain is (accurately) approximated as constant leading to vanishing strain gradients. We need another innovative experimental designs, where not only the strain but also the strain gradient is steered or directly measured.

4 Conclusion

We have presented the variational formulation for the generalized mechanics and applied this strategy for elastic and viscoelastic metamaterials. In the length scale of the metamaterial, additional parameters appear and their determination is yet unclear. These additional parameters are proposed in a formal manner by using the mathematical description. As a concrete example, we are convinced of the form of the kinetic energy such that in the case of metamaterial an analog term is proposed. Although formally correct, this term is difficult to understand and even detect by a measurement. By exploiting the variation formulation, we reach a very useful interpretation for this term being tantamount to a parameter in the viscoelastic material. Owing to this description, it is possible to suggest an experimental procedure to determine this specific parameter. The ultimate aim is to extract all parameters from one or several experiments. This task is conceivably very challenging and not yet resolved, which we hope to change in our future research.

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